MTH 225: Homework #9

Due Date: April 19, 2024

1. Consider the following quadratic form of real variables $\vec{x} = (x_1, x_2, x_3)$:

$$Q(\vec{x}) = Q(x_1, x_2, x_3) = 6x_1^2 - 4x_1x_2 - 2x_1x_3 + 6x_2^2 - 2x_2x_3 + 5x_3^2.$$

   (a) Express $Q(\vec{x})$ as $\vec{x}^T A\vec{x}$ for a symmetric matrix A.

   (b) Diagonalize Q (which is equivalent to diagonalizing A) by finding new variables y_1, y_2 and y_3 (express each in terms of x_1, x_2 and x_3), so that there are constants $a, b, c \in \mathbb{R}$ with $Q(x_1, x_2, x_3) = ay_1^2 + by_2^2 + cy_3^2$.

2. Consider the following quadratic form of real variables $\vec{x} = (x_1, x_2)$:

$$Q(\vec{x}) = Q(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 8x_2^2.$$

   (a) Express $Q(\vec{x})$ as $\vec{x}^T A\vec{x}$ for a symmetric matrix A.

   (b) Diagonalize Q and use it to graph the ellipse

$$5x^2 + 4xy + 8y^2 = 1.$$

3. For what values of $a, b, c \in \mathbb{R}$ is the the quadratic form

$$q(x, y, z) = x^2 + axy + y^2 + bxz + cyz + z^2$$

   nonnegative for all values of $(x, y, z) \in \mathbb{R}^3$.

4. (a) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $\det(A) > 0$.

   (b) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $\mathrm{Tr}(A) > 0$.

   (c) Prove that if $A \in M_{2\times 2}(\mathbb{C})$ is Hermitian and satisfies $\mathrm{Tr}(A) > 0$ and $\det(A) > 0$ then A is positive definite.

   (d) Find a symmetric matrix $A \in M_{3\times 3}(\mathbb{C})$ with positive determinant and positive trace that is not positive definite.

5. Show that $A \in M_{n\times n}$ and $A^T \in M_{n\times n}$ have the same characteristic polynomial.

6. Show that $A \in M_{n\times n}$ and $A^T \in M_{n\times n}$ have the same eigenvalues. Find a counterexample to show that they do not necessarily have the same eigenvectors.

7. The refined Gershgorin domain is given by $D_A^* = D_{A^T} \cap D_A$. Show that the eigenvalues of A must lie in the refined Gershgorin domain.

8. Show that if A is Hermitian, strictly diagonally dominant, and each diagonal entry is positive, then A is positive defininite.

9. Find an invertible matrix $A \in M_{2\times 2}(\mathbb{C})$ whose Gershgorin domain contains 0.

10. For each of the following matrices, (i) find the Gershgorin disks of A and A^T, (iI) plot the refined Gershgorin domain in the complex plane, (iii) compute the eigenvalues and confirm the truth of the Gershgorin circle theorem.

$$A = \begin{bmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Homework #9

#1.

Consider the following quadratic form

$$Q(\vec{x}) = 6x_1^2 - 4x_1x_2 - 2x_1x_3 + 6x_2^2 - 2x_2x_3 + 5x_3^2$$

(a) Express $Q(\vec{x})$ as $\vec{x}^T A \vec{x}$ for a symmetric matrix A.

(b) Diagonalize Q by finding new variables y_1, y_2, y_3 so that $Q(x_1, x_2, x_3) = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2$.

Solution:

(a) If $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$ it follows that $Q(\vec{x}) = \vec{x}^T A \vec{x}$.

(b)
$$\begin{aligned}
\det(\lambda I - A) &= \det\left(\begin{bmatrix} \lambda-6 & 2 & 1 \\ 2 & \lambda-6 & 1 \\ 1 & 1 & \lambda-5 \end{bmatrix}\right) \\
&= (\lambda-6)((\lambda-6)(\lambda-5)-1) - 2(2(\lambda-5)-1) + 2 - \lambda + 6 \\
&= (\lambda-6)(\lambda^2 - 11\lambda + 29) - 2(2\lambda - 11) + 2 - \lambda + 6 \\
&= (\lambda-6)(\lambda^2 - 11\lambda + 29) - 4\lambda + 22 + 2 - \lambda + 6 \\
&= (\lambda-6)(\lambda^2 - 11\lambda + 29) + 30 - 5\lambda \\
&= \lambda^3 - 11\lambda^2 + 29\lambda - 6\lambda^2 + 66\lambda - 6 \cdot 29 + 30 - 5\lambda \\
&= \lambda^3 - 17\lambda^2 + 90\lambda - 144 \\
&= p(\lambda)
\end{aligned}$$

Now,
$$\begin{aligned}
p(3) &= 3^3 - 17 \cdot 3^2 + 10 \cdot 3^3 - 16 \cdot 3^2 \\
&= 9(3 - 17 + 30 - 16) \\
&= 0.
\end{aligned}$$

There $\lambda_1 = 3$ is an eigenvalue. We now divide the polynomial

$$
\begin{array}{r|l}
 & \lambda^2-14\lambda+48 \\
\lambda-3 & \lambda^3-17\cdot\lambda^2+90\lambda-144 \\
 & -(\lambda^3-3\lambda^2) \\
\hline
 & -14\lambda^2+90\lambda \\
 & +(14\lambda-42\lambda) \\
\hline
 & 48\lambda-144
\end{array}
$$

Therefore,

$$p(\lambda)=(\lambda-3)(\lambda^2-14\lambda+48)$$
$$=(\lambda-3)(\lambda-6)(\lambda-8)$$

Consequently,

$$\lambda=3,6,8$$

$\underline{\lambda=3:}$

$$\lambda I-A=\begin{bmatrix}-3&2&1\\2&-3&1\\1&1&-2\end{bmatrix}\begin{matrix}+3R2\\-2R3\\ \end{matrix}\Rightarrow\begin{bmatrix}0&5&-5\\0&-5&5\\1&1&-2\end{bmatrix}\begin{matrix}/5\\+R1\\ \end{matrix}\Rightarrow\begin{bmatrix}0&1&-1\\0&0&0\\1&1&-2\end{bmatrix}\begin{matrix}\\ \\-R1\end{matrix}\Rightarrow\begin{bmatrix}0&1&-1\\0&0&0\\1&0&-1\end{bmatrix}$$

$$\mathrm{Ker}(-\lambda I-A)=\mathrm{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}=\mathrm{span}\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right\}$$

$\underline{\lambda=6:}$

$$\lambda I-A=\begin{bmatrix}0&2&1\\2&0&1\\1&1&1\end{bmatrix}\begin{matrix}\\-2R3\\ \end{matrix}\Rightarrow\begin{bmatrix}0&2&1\\0&-2&-1\\1&1&1\end{bmatrix}\begin{matrix}\\+R1\\-\frac{1}{2}R1\end{matrix}\Rightarrow\begin{bmatrix}0&2&1\\0&0&0\\1&0&\frac{1}{2}\end{bmatrix}\begin{matrix}/2\\ \\ \end{matrix}\Rightarrow\begin{bmatrix}0&1&\frac{1}{2}\\0&0&0\\1&0&\frac{1}{2}\end{bmatrix}$$

$$\Rightarrow\mathrm{Ker}(\lambda I-A)=\mathrm{span}\left\{\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\}=\mathrm{span}\left\{\begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}\right\}$$

$\underline{\lambda=8:}$

$$\lambda I-A=\begin{bmatrix}2&2&1\\2&2&1\\1&1&3\end{bmatrix}\begin{matrix}-2R3\\-R1\\ \end{matrix}\Rightarrow\begin{bmatrix}0&0&-5\\0&0&0\\1&1&3\end{bmatrix}\Rightarrow\begin{bmatrix}0&0&1\\0&0&0\\1&1&0\end{bmatrix}$$

$$\Rightarrow\mathrm{Ker}(\lambda I-A)=\mathrm{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}=\mathrm{span}\left\{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\right\}$$

Therefore,

$$U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$$

$$\Rightarrow U^* = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

Consequently,

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = U^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow y_1 = 1/\sqrt{3}\, x_1 + 1/\sqrt{3}\, x_2 + 1/\sqrt{3}\, x_3$$
$$y_2 = -1/\sqrt{6}\, x_1 - 1/\sqrt{6}\, x_2 + 2/\sqrt{6}\, x_3$$
$$y_3 = 1/\sqrt{2}\, x_1 - 1/\sqrt{2}\, x_2$$

$$\Rightarrow Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$
$$= (x_1 + x_2 + x_3)^2 + (x_1 + x_2 - 2x_3)^2 + 4(x_1 - x_2)^2$$

#2.

Consider the following quadratic form of real variables $\vec{x} = (x_1, x_2)$:

$$Q(\vec{x}) = 5x_1^2 + 4x_1x_2 + 8x_2^2$$

(a) Express $Q(\vec{x})$ as $\vec{x}^T A \vec{x}$ for a symmetric matrix A.

(b) Diagonalize Q and use it to graph the ellipse

$$5x^2 + 4xy + 8y^2 = 1.$$

Solution:

(a) $A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$.

(b) $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-5 & -2 \\ -2 & \lambda-8 \end{bmatrix}\right)$

$$= \lambda^2 - 13\lambda + 40 - 4$$
$$= \lambda^2 - 13\lambda + 36$$
$$= (\lambda - 4)(\lambda - 9)$$

Therefore, the eigenvalues are $\lambda=4$ and $\lambda=9$.

$\underline{\lambda=4}$:

$$\lambda I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \Rightarrow \ker(\lambda I - A) = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\right\}.$$

$\underline{\lambda=9}$:

$$\lambda I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \Rightarrow \ker(\lambda I - A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right\}.$$

Consequently,

$$U = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \Rightarrow U^* = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

and thus

$$y_1 = -2/\sqrt{5}\, x_1 + 1/\sqrt{5}\, x_2,$$
$$y_2 = 1/\sqrt{5}\, x_1 + 2/\sqrt{5}\, x_2$$

Therefore,

$$Q(\vec{x}) = 4y_1^2 + 9y_2^2$$
$$= \frac{4}{5}(-2x_1 + x_2)^2 + \frac{9}{5}(x_1 + 2x_2)^2$$

$$\Rightarrow 5x^2 + 4xy + 8y^2 = \frac{4}{5}(-2x+y)^2 + \frac{9}{5}(x+2y)^2 = 1.$$

$\underline{y=2x}$:

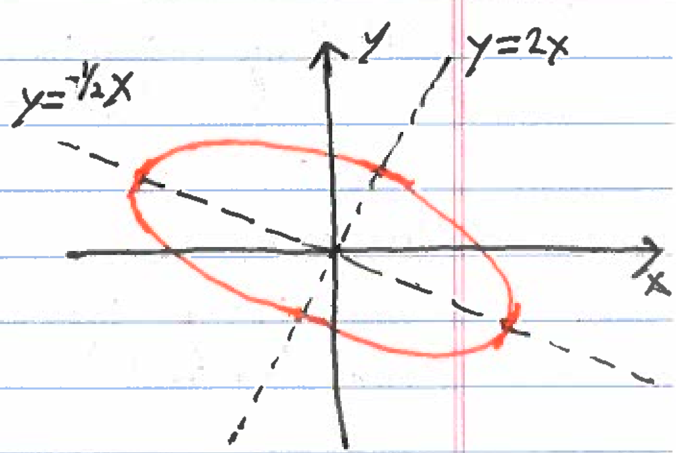
$$\frac{9}{5}(5x)^2 = 1$$

$$\Rightarrow x = 1/3\sqrt{5}$$
$$y = 2/3\sqrt{5}$$
$$\Downarrow$$
$$x^2 + y^2 = \frac{1}{45} \cdot 5$$
$$= 1/9$$
$$\Rightarrow \sqrt{x^2+y^2} = 1/3$$

$\underline{x=-2y}$:

$$\frac{4}{5}(5y)^2 = 1$$

$$\Rightarrow y = 1/2\sqrt{5}$$
$$x = -1/\sqrt{5}$$
$$\Downarrow$$
$$x^2 + y^2 = \frac{1}{5}\left(\frac{1}{4} + 1\right)$$
$$= 1/4$$
$$\sqrt{x^2+y^2} = 1/2$$



#3

For what values of $a,b,c\in\mathbb{R}$ is the quadratic form

$$q(x,y,z)=x^2+axy+y^2+bxz+cyz+z^2$$

nonnegative for all values of $(x,y,z)\in\mathbb{R}^3$.

Solution

Let $a'=a/2$, $b'=b/2$, and $c'=c/2$. The associated matrix is given by

$$A=\begin{bmatrix}1 & a' & b'\\ a' & 1 & c'\\ b' & c' & 1\end{bmatrix}$$

$$\det(A-\lambda I)=\det\left(\begin{bmatrix}1-\lambda & a' & b'\\ a' & 1-\lambda & c'\\ b' & c' & 1-\lambda\end{bmatrix}\right)$$

$$=(1-\lambda)((1-\lambda)^2-c'^2)-a'(a'(1-\lambda)-c'b')+b'(a'c'-b'(1-\lambda))$$

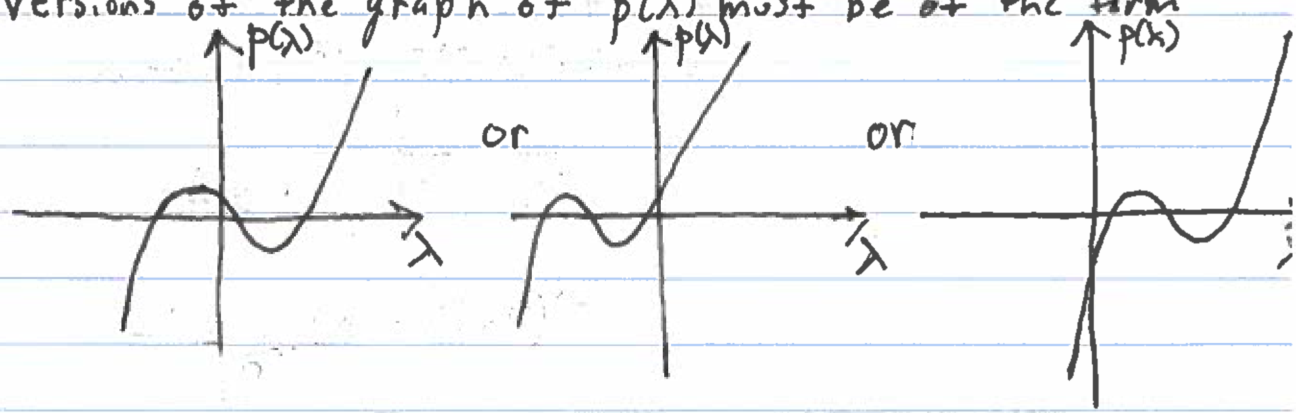
$$=(1-\lambda)(1-2\lambda+\lambda^2-c'^2)-a'^2(1-\lambda)+a'c'b'+b'a'c'-b'^2(1-\lambda)$$

$$=1-2\lambda+\lambda^2-c'^2-\lambda+2\lambda^2-\lambda^3+\lambda c'^2-a'^2+a'^2\lambda$$
$$+2a'c'b'-b'^2+b'^2\lambda$$

The characteristic polynomial is therefore given by

$$p(\lambda)=\lambda^3-3\lambda^2+(3-a'^2-b'^2-c'^2)\lambda+a'^2+b'^2+c'^2-2a'c'b'-1.$$

We need to determine qualitative features of the graph. We know that $\lim_{\lambda\to\infty}p(\lambda)=\infty$ and since A is Hermitian that all of the roots are real. Consequently, generic versions of the graph of $p(\lambda)$ must be of the form



The condition that all of the roots are positive are $p(0)<0$ and its first critical point is greater than 0. Computing, we have

$$p'(\lambda) = 3\lambda^2 - 6\lambda + (3 - a'^2 - b'^2 - c'^2)$$

Therefore, $p'(\lambda) = 0$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36 - 12(3 - a'^2 - b'^2 - c'^2)}}{6}$$

$$= 1 \pm \frac{\sqrt{12(a'^2 + b'^2 + c'^2)}}{6}$$

$$= 1 \pm \frac{\sqrt{a'^2 + b'^2 + c'^2}}{\sqrt{3}}$$

Therefore, for one condition we need

$$1 - \frac{\sqrt{a'^2 + b'^2 + c'^2}}{\sqrt{3}} > 0$$

$$\Rightarrow \sqrt{a'^2 + b'^2 + c'^2} < \sqrt{3}$$

$$\Rightarrow a'^2 + b'^2 + c'^2 < 3.$$

Now,

$$p(0) = a'^2 + b'^2 + c'^2 - 2a'c'b' - 1 < 0$$

$$\Rightarrow a'^2 + b'^2 + c'^2 < 2a'c'b' + 1$$

The two conditions are therefore

$$a'^2 + b'^2 + c'^2 < 3$$

$$a'^2 + b'^2 + c'^2 < 2a'c'b' + 1$$

#4

(a) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $\det(A) > 0$.

(b) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is positive definite then $Tr(A) > 0$.

(c) Prove that if $A \in M_{2\times 2}(\mathbb{C})$ is Hermitian and satisfies $Tr(A) > 0$ and $Det(A) > 0$ then A is positive definite.

(d) Find a symmetric matrix $A \in M_{3\times 3}(\mathbb{C})$ with positive determinant and positive trace that is not positive definite.

Solution:

(a-b) Since A is positive definite all of its eigenvalues are positive and thus $Det(A) = \lambda_i \cdot \ldots \cdot \lambda_n > 0$ and $Tr(A) = \lambda_1 + \ldots + \lambda_n > 0$.

(c) Since A is Hermitian it follows that $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore, since $Det(A) = \lambda_1 \lambda_2$ it follows that λ_1, λ_2 have the same sign. Finally since $Tr(A) = \lambda_1 + \lambda_2 > 0$ it follows that $\lambda_1, \lambda_2 \geq 0$ and thus A is positive semidefinite.

(d) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

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#5

Show that $A \in M_{n\times n}(\mathbb{C})$ and $A^T \in M_{n\times n}(\mathbb{C})$ have the same characteristic polynomial.

Solution:

$\det(\lambda I - A^T) = \det(\lambda I^T - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A)$.

■

#6

Show that A and A^T have the same eigenvalues. Find a counterexamp to show that they do not necessarily have the same eigenvectors.

Solution:

Since A and A^T have the same characteristic polynomials they have the same eigenvalues. However if we let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

it follows that the eigenvectors of A are given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

However,

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

which has eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\blacksquare$

#7

The refined Gershgorin domain is given by $D_A^* = D_{A^T} \cap D_A$. Show that the eigenvalues must lie in D_A^*.

Solution:

Since the eigenvalues of A are the same as A^T it follows that $\lambda \in D_{A^T}$ and $\lambda \in D_A \Rightarrow \lambda \in D_{A^T} \cap D_A$.

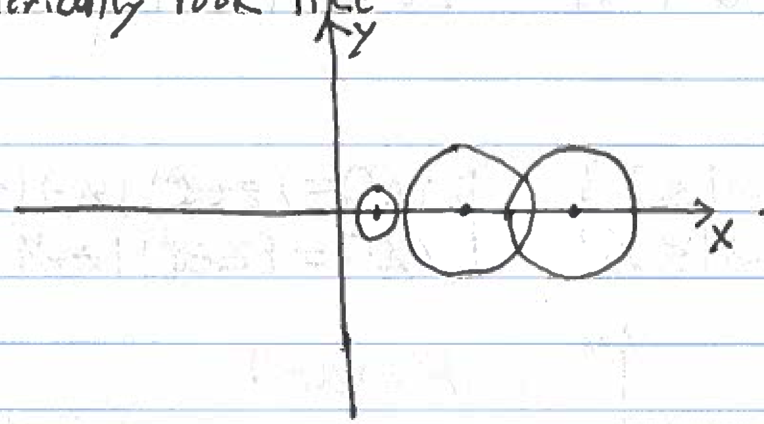
$\blacksquare$

#8

Show that if A is Hermitian, strictly diagonally dominant, and each diagonal entry is positive, then A is positive definite.

Solution:

Since A is Hermitian then its eigenvalues are real. Since each diagonal entry is positive and the matrix is strictly diagonally dominant it follows that the Gershgorin disks generically look like



Moreover, since the eigenvalues are real it follows they are positive and thus A is positive definite.

■

#9

Find an invertible matrix $A \in M_{2\times 2}(\mathbb{C})$ whose Gershgorin domain contains 0.

Solution:

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

A is invertible since $\det(A) = 1$. But,

$$0 \in D_1 = \{ z \in \mathbb{C} : |z-1| < 10 \}.$$

#10

For each of the following matrices

(i) Find the Gershgorin disks of A and A^T.

(ii) Plot the refined Gershgorin domain in the complex plane.

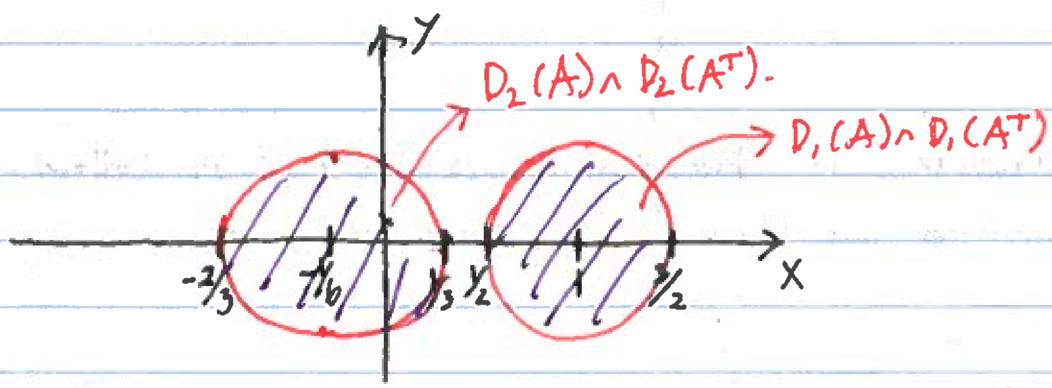
(iii) Compute the eigenvalues and confirm the truth of the Gershgorin circle theorem.

$$A=\begin{bmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix},\quad B=\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix},\quad C=\begin{bmatrix} -1 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{bmatrix},\quad D=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution:

(a) $D_1(A)=\{z\in\mathbb{C} : |z-1|\le \frac{2}{3}\}$, $D_1(A^T)=\{z\in\mathbb{C} : |z-1|\le\frac{1}{2}\}$

$D_2(A)=\{z\in\mathbb{C} : |z+\frac{1}{6}|\le\frac{1}{2}\}$, $D_2(A^T)=\{z\in\mathbb{C} : |z+\frac{1}{6}|\le\frac{2}{3}\}$

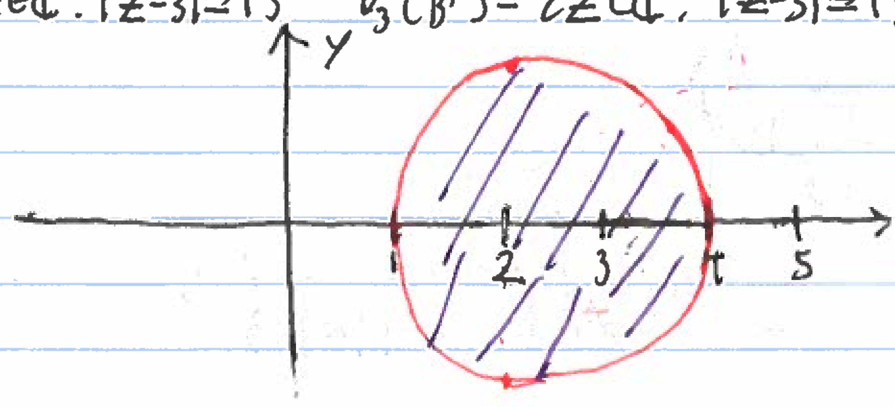


The shaded region is the refined Gershgorin regions

(b) $D_1(B)=\{z\in\mathbb{C} : |z-3|\le 1\}$, $D_1(B^T)=\{z\in\mathbb{C} : |z-3|\le 1\}$

$D_2(B)=\{z\in\mathbb{C} : |z-2|\le 2\}$, $D_2(B^T)=\{z\in\mathbb{C} : |z-2|\le 2\}$

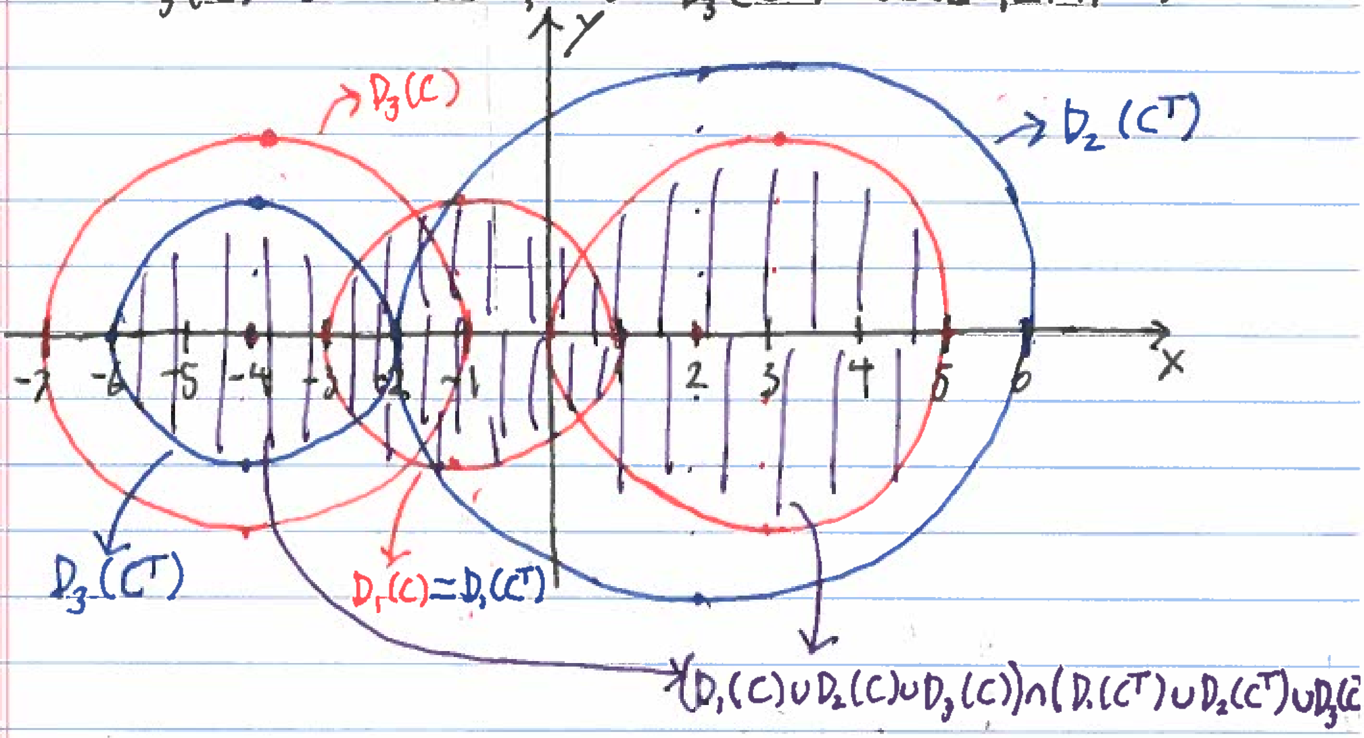
$D_3(B)=\{z\in\mathbb{C} : |z-3|\le 1\}$, $D_3(B^T)=\{z\in\mathbb{C} : |z-3|\le 1\}$.



(c) $D_1(C)=\{z\in\mathbb{C}: |z+1|\le 2\}$, $D_1(C^T)=\{z\in\mathbb{C}: |z+1|\le 2\}$

$D_2(C)=\{z\in\mathbb{C}: |z-2|\le 3\}$, $D_2(C^T)=\{z\in\mathbb{C}: |z-2|\le 4\}$

$D_3(C)=\{z\in\mathbb{C}: |z+4|\le 3\}$ $D_3(C^T)=\{z\in\mathbb{C}: |z+4|\le 2\}$



The shaded region is the refined Gershgorin region.

(d) $D_1(D)=\{z\in\mathbb{C}: |z|\le 1\}$, $D_1(D^T)=\{z\in\mathbb{C}: |z|\le 0\}$

$D_2(D)=\{z\in\mathbb{C}: |z-1|\le 1$, $D_2(D^T)=\{z\in\mathbb{C}: |z-1|\le 2\}$

$D_3(D)=\{z\in\mathbb{C}: |z-1|\le 1\}$ $D_3(D^T)=\{z\in\mathbb{C}: |z-1|\le 1$