MTH 225: Homework #1

Due Date: January 26, 2024

1. Sign up for Piazza. I will check the roster for your name.

2. Solve the following systems of equations using row reduction. For those systems with infinitely many solutions, find a parametric description of the solution space.

(a)

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v + w &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y &= 7 \\ x - 3y &= 5 \\ 5x + 3y &= 18 \end{aligned}$$

(c)

$$\begin{aligned} x - z &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

3. Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ under the standard operations of vector addition and scalar multiplication. If a set is a subspace, prove it. If a set is not a subspace explain why.

   (a) $V = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$
   (b) $W = \{(x, y, z) \in \mathbb{R}^3 : y = 1\}$
   (c) $U = \{(x, y, z) \in \mathbb{R}^3 : xy = 0\}$
   (d) $S = \{(x, y, z) \in \mathbb{R}^3 : z - x = 2y\}$

4. Show that the set of 2×2 matrices of the following form is a subspace of $M_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} \text{ for } a, b, c \in \mathbb{R}.$$

5. Determine whether the following sets are subspaces of $P_2(\mathbb{R})$. If a set is a subspace prove it. If a set is not a subspace explain why it is not.

   (a) Polynomials of the form $p(t) = at^2$ for $a \in \mathbb{R}$.
   (b) Polynomials of the form $p(t) = a + t^2$ for $a \in \mathbb{R}$.
   (c) Polynomials $p(t)$ in $P_2(\mathbb{R})$ with integer coefficients.
   (d) Polynomials $p(t)$ in $P_2(\mathbb{R})$ with $p(0) = 0$.

6. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?

7. Find a vector that, together with the vectors $[1, 1, 1]^T$ and $[1, 2, 1]^T$, forms a basis of $\mathbb{R}^3$.

8. Determine if the following sets of vectors are linearly independent and determine a basis for its span.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

9. Suppose that W_1 and W_2 are subspaces of $\mathbb{R}^n$. Define the following subsets of $\mathbb{R}^n$:

$$\begin{aligned} W_1 + W_2 &= \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\} \\ W_1 \cap W_2 &= \{w \in \mathbb{R}^n \mid w \in W_1 \text{ and } w \in W_2\} \\ W_1 \cup W_2 &= \{w \in \mathbb{R}^n \mid w \in W_1 \text{ or } w \in W_2\} \end{aligned}$$

(a) Prove that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

(b) Prove that $W_1 + W_2$ is a subspace of $\mathbb{R}^n$.

(c) Show by means of an example that $W_1 \cup W_2$ is not necessarily a subspace of $\mathbb{R}^n$. Which of the three rules are not satisfied?

(d) Show that W_1 and W_2 are subsets of $W_1 + W_2$.

(e) Show that if V is another subspace that contains both W_1 and W_2, then V contains $W_1 + W_2$.