MTH 225: Homework #10

Due Date: April 26, 2024

1. For each of the following matrices, first determine if it is a transition matrix. If it is a transition matrix, sketch the associated graph along with the edges labeled by their associated probabilities and determine if the matrix is regular. If the matrix is regular, find the stationary probability vector.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{3} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{4}{5} \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} .3 & .3 & .2 \\ .3 & .2 & .5 \\ .4 & .3 & .3 \end{bmatrix}, \quad G = \begin{bmatrix} .1 & .5 & .4 \\ .6 & .1 & .3 \\ .3 & 0 & .7 \end{bmatrix}, \quad H = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \end{bmatrix},$$

$$J = \begin{bmatrix} 0 & .2 & 0 & 1 \\ .5 & 0 & .3 & 0 \\ 0 & .8 & 0 & 0 \\ .5 & 0 & .7 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} .1 & .2 & .3 & .4 \\ .2 & .5 & .3 & .1 \\ .3 & .3 & .1 & .3 \\ .4 & .1 & .3 & .2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & .6 & 0 & .4 \\ .5 & 0 & .3 & .1 \\ 0 & .4 & 0 & .5 \\ .5 & 0 & .7 & 0 \end{bmatrix}.$$

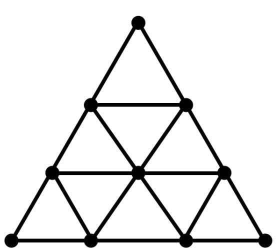
2. A business executive is managing three branches, labeled A, B, and C, of a corporation. They never visit the same branch on consecutive days. If they visit branch A one day, they visit branch B the next day. If they visit either branch B or C that day, then the next day the are twice as likely to visit branch A as to visit branch B or C. Explain why the resulting transition matrix is regular. Which branch do they visit the most often in the long run?

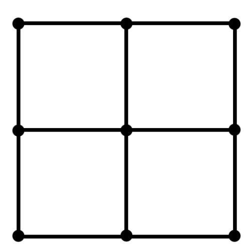
3. A traveling salesman visits the three cities of Atlanta, Boston, and Chicago. The matrix

$$T = \begin{bmatrix} 0 & .5 & .5 \\ 1 & 0 & .5 \\ 0 & .5 & 0 \end{bmatrix}$$

describes the transition probabilities of their trip. Describe their travels in words, and calculate how they visit each city on average.

4. Suppose an insect crawls along the edges of the graphs drawn below. Upon arriving at a vertex, there is an equal probability of choosing any edge to leave the vertex. For each graph, set up the Markhov chain described by the insect's motion, and determine how often, on average, it visits each vertex.





5. Let T be a regular transition matrix with corresponding stationary probability vector $\vec{u}^*$.
   (a) Prove that $\lim_{k\to\infty} T^k = P = [\vec{u}^*|\vec{u}^*|\cdots|\vec{u}^*]$, i.e., P is the matrix with every column equal to $\vec{u}^*$. **Hint:** This about how you would compute each column of T^k and then take the limit.
   (b) Explain why $P\vec{u}^* = \vec{u}^*$.
   (c) Prove that P satisfies $P^2 = P$.
   (d) Find $\lim\limits_{k\to\infty} T^k$ when
$$T = \begin{bmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{bmatrix}$$
6. Prove that, for all $0 \le p, q \le 1$ with $p + q > 0$, the stationary probability vector of the transition matrix
$$T = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$$
is $\vec{u}^*$ is $\vec{u}^* = \left[\frac{q}{p+q}, \frac{p}{p+q}\right]^T$.
7. Let T be a transition matrix. Prove that if $\vec{u}$ is a probability vector, then so is $\vec{v} = T\vec{u}$.
8. Prove that if T and S are transition matrices, then so is their product TS.
9. Prove that if T is a transition matrix, then so is T^k for all $k \in \mathbb{N}$.