MTH 225: Homework #2

Due Date: February 02, 2024

1. Let $F(a,b)$ be the vector space of all real valued function on the interval (a,b) over the field of real numbers with the normal operations of addition and scalar multiplication. Determine if the following subsets of $F(a,b)$ are a subspace of $F(a,b)$. If they are not a subspace explain why and if they are a subspace prove it.
   (a) All functions f in $F(a,b)$ for which $f(a) = 0$.
   (b) All functions f in $F(a,b)$ for which $f(a) = 1$.
   (c) All continuous functions f in $F(a,b)$ for which $\int_a^b f(x)dx = 0$.
   (d) All differentiable functions f in $F(a,b)$ for which $f'(x) = f(x)$.

2. Show that
$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right\}$$
forms a basis for $M_{2\times 2}(\mathbb{R})$.

3. Consider the map $T : P_3(\mathbb{R}) \mapsto \mathbb{R}^4$ defined by
$$T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_1, a_1, a_1, a_1).$$
   (a) Show that T is a linear transformation.
   (b) Find bases for $\ker(T)$, $\mathrm{im}(T)$ and verify the Rank + Nullity Theorem.

4. Explain why there is no linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ that satisfies the following
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 15 \\ 6 \\ 5 \end{bmatrix}.$$

5. Define the linear transformation $T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{R}^3$ by
$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a+b \\ b+c \\ a+b \end{bmatrix}.$$
   (a) Find a basis for the kernel of T.
   (b) Find a basis for the range of T.

6. The trace of an $n \times n$ matrix is the sum of the diagonal entries:

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

   (a) Show that $T : M_{3\times 3}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $T(A) = \text{trace}(A)$ is a linear transformation.

   (b) Find the dimension and a basis for $\ker(T)$ and generalize to $n \times n$ matrices.

7. Let $A = (a_{ij})$ be an $n \times n$ matrix whose i, j th entry is a_{ij}. Define the transpose of A as the matrix whose ij entry is a_{ji}, i.e. $A^t = (a_{ji})$. A matrix is called *symmetric* if $A = A^t$.

   (a) Prove that the symmetric $n \times n$ matrices form a subspace of $M_{n\times n}(\mathbb{R})$.

   (b) Find the dimension of the subspace of symmetric $n \times n$ matrices. Prove your answer.