MTH 225: Homework #3

Due Date: February 09, 2024

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation with values

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix},\ \text{and } T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}).$$

   (a) Find the matrix $[T(\mathcal{B}, \mathcal{S}_2)]$ of T in the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

   where $[T(\mathcal{B}, \mathcal{S}_2)][\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{S}_2}$.

   (b) Find the matrix $[T(\mathcal{S}_1, \mathcal{S}_2)]$ of T in the standard bases

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

   where $[T(\mathcal{S}_1, \mathcal{S}_2)][\mathbf{v}]_{\mathcal{S}_1} = [T(\mathbf{v})]_{\mathcal{S}_2}$.

2. Consider the two bases $\mathcal{B}$ and $\mathcal{S}$ of $\mathbb{R}^2$ where

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \text{ and } \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 3x + y \end{bmatrix}.$$

   (a) Find the matrix representation of the identity linear transformation $I : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the input basis $\mathcal{B}$ and output basis $\mathcal{S}$. That is, find the matrix $P = [I(\mathcal{B}, \mathcal{S})]$ where $P[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{S}}$.

   (b) Find the matrix representation of the identity linear transformation $I : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the input basis $\mathcal{S}$ and output basis $\mathcal{B}$. That is, find the matrix $Q = [I(\mathcal{S}, \mathcal{B})]$ where $Q[\mathbf{v}]_{\mathcal{S}} = [\mathbf{v}]_{\mathcal{B}}$.

   (c) Find the matrix $A = [T(\mathcal{S}, \mathcal{S})]$, where $A[\mathbf{v}]_{\mathcal{S}} = [T(\mathbf{v})]_{\mathcal{S}}$.

   (d) Find the matrix $B = [T(\mathcal{S}, \mathcal{B})]$, where $B[\mathbf{v}]_{\mathcal{S}} = [T(\mathbf{v})]_{\mathcal{B}}$.

   (e) Write B as a product of A and any of the matrices P and Q that are relevant.

3. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(a + bx + cx^2) = (a + c) + bx^2$. Let $\mathcal{S} = \{1, x, x^2\}$ be the standard basis of $P_2(\mathbb{R})$, and let $\mathcal{B} = \{1 + x, x, 1 + x^2\}$ be another basis.

   (a) Find the matrix $[T(\mathcal{S}, \mathcal{S})]$ of T with respect to $\mathcal{S}$.

   (b) Find the matrix $[T(\mathcal{B}, \mathcal{B})]$ of T with respect to $\mathcal{B}$.

   (c) Find an invertible matrix P so that $P[T(\mathcal{B}, \mathcal{B})]P^{-1} = [T(\mathcal{S}, \mathcal{S})]$.

4. Suppose $\mathbf{v} \in V$ is a vector in some vector space V and $T : V \to V$ a linear transformation such that $\mathcal{B} = \{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots, T^{n-1}(\mathbf{v})\}$ is a basis for V.

   (a) Show that there exists constants $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that

   $$T^n(\mathbf{v}) = a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{n-1}T^{n-1}(\mathbf{v}).$$

   (b) Find the matrix $[T(\mathcal{B}, \mathcal{B})]$ of T with respect to $\mathcal{B}$.

   (c) When does T map onto V and when is T one-to-one?

   (d) Find the characteristic polynomial $c_T(x)$ of T. (Hint: Do cases $n = 2$ and 3 to get a conjecture and prove it by induction).

5. Suppose that the vector $v = (1, 1) \in \mathbb{R}^2$ is an eigenvector of $A \in \mathrm{Mat}_{2\times2}(\mathbb{R})$ corresponding to the eigenvector λ. Draw on a graph the vectors v and Av for each of the following cases. (Make a separate graph for each part.)

   (a) $\lambda > 1$.

   (b) $\lambda = 1$.

   (c) $0 < \lambda < 1$.

   (d) $\lambda = 0$.

   (e) $-1 < \lambda < 0$.

   (f) $\lambda = -1$.

   (g) $\lambda < -1$.

6. Prove that 0 is an eigenvalue of T if and only if $Ker(T)$, the nullspace of T, is $\neq \{0\}$.

7. If λ is an eigenvalue for an $n \times n$ matrix A, show that λ^2 is an eigenvalue for A^2. More generally prove that if $f(x)$ is any polynomial with coefficients in $\mathbb{R}$, then $f(\lambda)$ is an eigenvalue for $f(A)$.

8. A matrix N is called *nilpotent* if $N^k = 0$ for some positive integer k. Prove that the only possible eigenvalue of a nilpotent matrix is 0.