MTH 225: Homework #5

Due Date: March 01, 2024

1. Suppose $A, B \in \mathbb{M}_{n\times n}(\mathbb{C})$ are two similar matrices. Prove that A and B have the same characteristic polynomial. **Hint:** For this problem you might have to review properties of determinants.
2. Prove that if $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable then A has a square root, i.e., there exists $B \in M_{n\times n}(\mathbb{C})$ such that $B^2 = A$.
3. Let A and $\vec{v}$ be given by
$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$
   (a) Find an orthonormal basis for $\mathrm{im}(A)$.
   (b) Find an orthonormal basis for $\mathrm{im}(A)^{\perp}$.
   (c) Compute the orthogonal projection of $\vec{v}$ onto $\mathrm{im}(A)$.
   (d) Compute the orthogonal projection of $\vec{v}$ onto $\mathrm{im}(A)^{\perp}$.
4. Let W be a subspace of $\mathbb{C}^n$ and $\{\vec{u}_1, \ldots \vec{u}_k\}$ be an orthonormal basis of W.
   (a) Show that for all $\vec{v} \in \mathbb{C}^n$ there exists $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^{\perp}$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$.
   (b) Prove that $W \cap W^{\perp} = \{0\}$.
   (c) Prove that $\dim(W) + \dim(W^{\perp}) = n$.
5. Prove that if S is a subset of $\mathbb{C}^n$ then $(S^{\perp})^{\perp} = \mathrm{span}(S)$
6. Let $A \in M_{m\times n}(\mathbb{C})$.
   (a) Prove that $\ker(A) \subseteq \ker(A^*A)$.
   (b) Prove that $\ker(A^*A) \subseteq \ker(A)$. **Hint:** If $\vec{v} \in \ker(A^*A)$, what is $\langle A\vec{v}, A\vec{v}\rangle$?
   (c) Prove that $\mathrm{rank}(A^*A) = \mathrm{rank}(A)$.
   (d) Prove that the columns of A are linearly independent if and only if A^*A is invertible.
7. Suppose $A \in M_{n\times n}(\mathbb{C})$ and $z, w \in \mathbb{C}$.
   (a) Prove that $\overline{zw} = \bar{z}\bar{w}$.
   (b) Prove that $\det(A) = \overline{\det(A^*)}$.
   (c) Prove that $|\det(A)|$ equals the product of its singular values.
8. Prove that if λ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.
9. Prove that if $A \in M_{m\times n}(\mathbb{C})$ is a rank 1 matrix then it is of the form $\vec{u}\vec{v}^*$ for some vectors $\vec{u}$ and $\vec{v}$.

10. Determine the singular value decompositions of the following matrices.

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(e) $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

11. If P is a unitary matrix, show that PA has the same singular values as A.