MTH 225: Homework #6

Due Date: March 08, 2024

1. Let $V = C^\infty([-1,1])$, i.e., the vector space of infinitely differentiable real valued functions defined on the interval $[-1,1]$. Define the operation $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

   and let $U \subset V$ be the subspace defined by $U = \text{span}\{1, x, x^2, x^3\}$.

   (a) Show that the operation $\langle \cdot, \cdot \rangle$ defined above is an inner product on V.

   (b) Using the above inner product, find an orthonormal basis for U.

2. Suppose $A, B \in M_{n\times n}(\mathbb{C})$ are unitary matrices. Prove that AB is a unitary matrix.

3. Suppose that $A = P\Sigma Q^*$ is a singular value decomposition of A. What is a singular value decomposition for A^*?

4. Suppose $A \in M_{n\times n}(\mathbb{C})$.

   (a) Prove that $\ker(A^*) = (\text{im}(A))^\perp$.

   (b) Prove that $\text{im}(A^*) = (\ker(A))^\perp$.

   (c) Prove that $\ker(A) = (\text{im}(A^*))^\perp$.

   (d) Prove that $\text{im}(A) = (\ker(A^*))^\perp$.

   (e) Let $\vec{b} \in \mathbb{C}^n$. Prove that there exists a $\vec{v} \in \mathbb{C}^n$ that satisfies $A\vec{v} = \vec{b}$ if and only if $\langle b, \vec{w} \rangle = 0$ for all $\vec{w}$ satisfying $A^*\vec{w} = 0$.

5. Suppose $A, B \in M_{n\times n}(\mathbb{C})$.

   (a) Prove that $A + A^*$ and $i(A - A^*)$ are Hermitian matrices.

   (b) Prove that A can be written as a linear combination of two Hermitian matrices. These two Hermitian matrices are sometimes called the real and imaginary components of a matrix.

   (c) Prove that if A and B are Hermitian then AB is Hermitian if and only if $AB = BA$, i.e., the matrices commute.

   (d) Prove that if A and B are Hermitian matrices then $AB+BA$ and $i(AB-BA)$ are also Hermitian matrices.

   (e) Prove that if A and B are Hermitian matrices then the matrix C defined by $AB - BA = iC$ is also Hermitian.

6. Suppose $A \in M_{n\times n}(\mathbb{C})$ is both Hermitian and unitary. Prove that its eigenvalues are all ± 1.

7. A matrix $A \in M_{n\times n}(\mathbb{C})$ is called **normal** if $AA^* = A^*A$.

   (a) Prove that Hermitian matrices and unitary matrices are normal matrices.

   (b) Show that the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

   is not a symmetric, Hermitian or unitary matrix, but is a normal matrix.

   (c) Show that A above has an orthogonal basis of eigenvectors. Are the eigenvalues real numbers?

   (d) Show that the following matrix is a normal matrix and has an orthogonal basis of eigenvectors

$$B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$