Lecture 18: How Do We Rank Things?

Simple Idea (Football)

$$S_i = \frac{1}{n_i} \sum_{j=1}^{N} a_{ij} r_j$$

r_j → ranking

S_i: Score of team i

n_i: number of games

a_{ij}: nonnegative matrix measuring outcome of game between team i and j.

- pick $a_{ij} = 1$ if team i beats team j.

Letting: $A_{ij} = \frac{1}{n_i} a_{ij}$ we have

$$\vec{S} = A\vec{r}.$$

We assume that the score should be proportional to $\vec{r}$

$$\Rightarrow A\vec{r} = \lambda \vec{r}$$

Therefore, the ranking should be an eigenvector with positive entries and non-negative entries.

Theorem (Perron-Frobenius Theorem) - If A is a regular matrix with non-negative entries then A has an eigenvector $\vec{v}^*$ with positive entries and a corresponding positive eigenvalue λ^*. Moreover, λ^* has geometric multiplicity one and if λ is another eigenvalue then $\lambda^* > |\lambda|$.

Power Method:

How do we find $\vec{v}^*$?

- Let $\{\vec{v}^*, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of eigenvectors with eigenvalues $\{\lambda^*, \lambda_2, \dots, \lambda_n\}$

- Let $\vec{v} = c_1 \vec{v}^* + \dots + c_n \vec{v}_n$.

$$\Rightarrow A\vec{v} = c_1 \lambda^* \vec{v}^* + \dots + c_n \lambda_n \vec{v}_n$$
$$\Rightarrow A^2\vec{v} = c_1 \lambda^{*2} \vec{v}^* + \dots + c_n \lambda_n^2 \vec{v}_n$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\Rightarrow A^k\vec{v} = c_1 (\lambda^*)^k \vec{v}^* + \dots + c_n \lambda_n^k \vec{v}_n$$

- If we normalize we have:

$$\frac{A^k\vec{v}}{\|A^k\vec{v}\|} = \frac{c_1(\lambda^*)^k\vec{v}^* + \dots + c_n(\lambda_n)^k\vec{v}_n}{\|c_1(\lambda^*)^k\vec{v}^* + \dots + c_n(\lambda_n)^k\vec{v}_n\|}$$
$$= \frac{\cancel{(\lambda^*)^k}}{\cancel{(\lambda^*)^k}} \frac{c_1\vec{v}^* + \dots + c_n(\lambda_n/\lambda^*)^k\vec{v}_n}{\|c_1\vec{v}^* + \dots + c_n(\lambda_n/\lambda^*)^k\vec{v}_n\|}$$

- Therefore,

$$\lim_{k\to\infty} \frac{A^k\vec{v}}{\|A^k\vec{v}\|} = \frac{\vec{v}^*}{\|\vec{v}^*\|}.$$