Lecture #3: Span and Linear Independence

Definition - A vector $\vec{w} \in V$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if it can be written in the form

$$\vec{w} = k_1\vec{v}_1 + \ldots + k_n\vec{v}_n$$

where $k_1, \ldots, k_n \in F$.

Example:

Let $\vec{u} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}6\\4\\2\end{bmatrix}$. Is $\vec{w} = \begin{bmatrix}9\\2\\7\end{bmatrix}$ a linear combination of $\vec{u}$ and $\vec{v}$?

$$\begin{bmatrix}9\\2\\7\end{bmatrix} = k_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + k_2\begin{bmatrix}6\\4\\2\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc:c}1 & 6 & 9\\2 & 4 & 2\\-1 & 2 & 7\end{array}\right]\begin{array}{l}\\-2R1\\+R1\end{array} \Rightarrow \left[\begin{array}{cc:c}1 & 6 & 9\\0 & -8 & -16\\0 & 8 & 16\end{array}\right] \Rightarrow \left[\begin{array}{cc:c}1 & 6 & 9\\0 & 1 & 2\\0 & 0 & 0\end{array}\right]$$

$$\Rightarrow k_2 = 2$$
$$k_1 = -3$$

$$\Rightarrow \vec{w} = -3\begin{bmatrix}1\\2\\-1\end{bmatrix} + 2\begin{bmatrix}6\\4\\2\end{bmatrix}$$

Definition - If V is a vector space and $\vec{v}_1, \ldots, \vec{v}_n$ are vectors in V, then the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$ is a subspace of V called $\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$.

Example:

Is $2x^2+2x+1 \in \text{span}\{x^2+x, x^2-1, x+1\}$?

$$\Rightarrow a(x^2+x) + b(x^2-x) + c(x+1) = 2x^2+2x+1.$$

$$\Rightarrow \begin{array}{l} a+b=2 \\ a+c=1 \\ c-b=1 \end{array} \Rightarrow \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

There is no solution so

$$2x^2+x+1 \notin \text{span}\{x^2+x,\ x^2-1,\ x+1\}.$$

Definition- A set of vectors $\vec{v}_1,\dots,\vec{v}_n \in V$ are linearly independent if the only solution to the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0$$

is $c_1 = c_2 = \dots = c_n = 0$.

Definition- A set of vectors $\vec{v}_1,\dots,\vec{v}_n \in V$ is a basis for V if

1. $\text{span}\{\vec{v}_1,\dots,\vec{v}_n\} = V$
2. $\vec{v}_1,\dots,\vec{v}_n$ are linearly independent.

The dimension of V is $\dim(V) = n$.

Example:

Find a basis for the subspace spanned by

$$\text{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 15 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix} \right\} = W$$

We need to determine if any vector can be written as a linear combination of another.

Row Reduction:

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \begin{array}{c} \\ +2R1 \\ \\ \\ -3R1 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 1 & -5 & -6 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \\ -R2 \\ -R2 \\ \\ \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{bmatrix}$$

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix} \right\}$$