Lecture 6: Coordinates and Change of Basis

Example:

Let $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \beta_1 = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \beta_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ -5x+13y \\ -7x+16y \end{bmatrix}$$

1. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is $[\vec{v}]_\beta$?

$$\Rightarrow \vec{v} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Coordinates

$$\Rightarrow \left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 2 & 1 \end{array}\right] \begin{matrix} -3R2 \\ \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 2 & 1 \end{array}\right] \begin{matrix} \\ +2R1 \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 0 & -1 & 2 \\ 1 & 0 & -3 \end{array}\right]$$

$\Rightarrow c_1 = -3$

$c_2 = 2$

$$\Rightarrow [\vec{v}]_{\beta_1} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

2. What is $[T(S_1, S_2)]$?

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -5 \\ -7 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 13 \\ 16 \end{bmatrix}$$

$$\Rightarrow [T(S_1, S_2)] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ 7 & 16 \end{bmatrix}$$

3. What is $[T(\beta_1, S_2)]$?

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}1\\-2\\-5\end{bmatrix},\ T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)=\begin{bmatrix}2\\1\\-3\end{bmatrix}$$

$$\Rightarrow [T(\beta_1, S_2)]=\begin{bmatrix}1&2\\-2&1\\-5&-3\end{bmatrix}$$

4. What is $[T(\beta_1, \beta_2)]$?

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}1\\-2\\-5\end{bmatrix},\ T\left(\begin{bmatrix}5\\2\end{bmatrix}\right)=\begin{bmatrix}2\\1\\-3\end{bmatrix}$$

$$\Rightarrow c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}-1\\2\\2\end{bmatrix}+c_3\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}1\\-2\\5\end{bmatrix},\begin{bmatrix}2\\1\\-3\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc:c:c}1&-1&0&1&2\\0&2&1&-2&1\\-1&2&2&-5&-3\end{array}\right]\begin{matrix}\\ \\ +R1\end{matrix} \Rightarrow \left[\begin{array}{ccc:c:c}1&-1&0&1&2\\0&2&1&-2&1\\0&1&2&-4&-1\end{array}\right] \updownarrow \left[\begin{array}{ccc:c:c}1&-1&0&1&2\\0&1&2&-4&-1\\0&2&1&-2&1\end{array}\right]\begin{matrix}+R2\\ \\ -2R2\end{matrix}$$

$$\Rightarrow \left[\begin{array}{ccc:c:c}1&0&2&-3&1\\0&1&2&-4&-1\\0&0&-3&6&3\end{array}\right]\begin{matrix}\\ \\ /-3\end{matrix} \Rightarrow \left[\begin{array}{ccc:c:c}1&0&2&-3&1\\0&1&2&-4&-1\\0&0&1&-2&-1\end{array}\right]\begin{matrix}-2R3\\ -2R3\\ \end{matrix} \Rightarrow \left[\begin{array}{ccc:c:c}1&0&0&1&3\\0&1&0&0&1\\0&0&1&-2&-1\end{array}\right]$$

$$\Rightarrow \begin{bmatrix}1\\-2\\-5\end{bmatrix}=1\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}+0\cdot\begin{bmatrix}-1\\2\\2\end{bmatrix}-2\begin{bmatrix}0\\1\\2\end{bmatrix} \Rightarrow \begin{bmatrix}1\\-2\\-5\end{bmatrix}=\begin{bmatrix}1\\0\\-2\end{bmatrix}_\beta$$

$$\begin{bmatrix}2\\1\\-3\end{bmatrix}=3\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}+1\cdot\begin{bmatrix}-1\\2\\2\end{bmatrix}-1\begin{bmatrix}0\\1\\2\end{bmatrix} \Rightarrow \begin{bmatrix}2\\1\\-3\end{bmatrix}=\begin{bmatrix}3\\1\\-1\end{bmatrix}_\beta$$

$$\Rightarrow [T(\beta_1, \beta_2)]=\begin{bmatrix}1&3\\0&1\\-2&-1\end{bmatrix}$$

Change of Basis Example:

Let $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

1. What is $P = [I(\beta, S_1)]$?

$$I\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad I\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

2. What is $Q = [I(S_1, \beta)]$?

$$I\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\beta}.$$

$$I\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\beta}.$$

$$\Rightarrow Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

3. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -2x+4y \end{bmatrix}$, what is $[T(S_1, S_1)]$?

$$\Rightarrow T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow [T(S_1, S_1)] = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

4. What is $[T(\beta, \beta)]$?

Let $D = [T(\beta, \beta)]$, $\tilde{T} = [T(S_1, S_1)]$

Therefore, for all $\vec{v} \in \mathbb{R}^2$:

$$D \cdot [v]_\beta = Q \cdot \tilde{T} \cdot P [v]_\beta$$

$$\Rightarrow D = Q \tilde{T} P$$

$$\Rightarrow D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$ (A diagonal matrix).