Lecture #7: Diagonalization

Example:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{v}) = A\vec{v}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow [T(S,S)] = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Consider the basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$$

$$\Rightarrow T(\vec{v}_1) = A\vec{v}_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\vec{v}_1$$

$$T(\vec{v}_2) = A\vec{v}_2 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \end{bmatrix} = 5\vec{v}_2$$

Therefore,

$$[T(\beta,\beta)] = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

Now,

$$\underbrace{[T(S,S)]}_{A}\vec{v} = \underbrace{[I(S,\beta)]}_{P^{-1}} \underbrace{[T(\beta,\beta)]}_{\Delta} \underbrace{[I(\beta,S)]}_{P}\vec{v}$$

$$\Rightarrow \boxed{A = P^{-1} \Delta P}$$

* Two matrices A, Δ are similar if there exists P such that $A = P^{-1}\Delta P$

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}, \quad P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}} \longrightarrow \text{Diagonalization.}$$

*The key to diagonalization was the existence of a vector $\vec{v}$ (eigenvector) and scalar $\lambda \in F$ (eigenvalue) such that

$$T(\vec{v}) = \lambda\vec{v}, \quad [T(S,S)]\vec{v} = \lambda\vec{v}$$

If $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of eigenvectors then

$$\boxed{[T(\beta,\beta)] = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}$$

How to find eigenvalues?

Theorem - $\lambda \in F$ is an eigenvalue of T if and only if

$$\det(\lambda I - T) = 0$$

~~prf~~:

λ is an eigenvalue of $T \Leftrightarrow$ there exists $0 \neq \vec{v}$ with $T(\vec{v}) = \lambda\vec{v}$ $\Leftrightarrow T(\vec{v}) - \lambda\vec{v} = 0 \Leftrightarrow (T - \lambda I)\vec{v} = 0 \Leftrightarrow \vec{v} \in \ker(T - \lambda I) \Leftrightarrow T - \lambda I$ is not invertible $\Leftrightarrow \det(T - \lambda I) = 0 \Leftrightarrow \det(\lambda I - T) = 0$.

*The function $c_T(\lambda) = \det(T - \lambda I)$ is called the characteristic polynomial

Definition - Let λ be an eigenvalue of T. The set E_λ of all vectors $\vec{v}$ with $A\vec{v} = \lambda\vec{v}$ is called the eigenspace of λ.

$\mathrm{Dim}(E_\lambda) =$ geometric multiplicity of λ.