MTH 225
Quiz #7

1. Suppose that $\vec{u}, \vec{v} \in \mathbb{C}^n$ are orthonormal and let

$$A = \vec{u}\vec{v}^* + \vec{v}\vec{u}^*.$$

(a) Show that A is a Hermitian matrix.

$$\begin{aligned} A^* &= (\vec{u}\vec{v}^* + \vec{v}\vec{u}^*)^* \\ &= (\vec{u}\vec{v}^*)^* + (\vec{v}\vec{u}^*)^* \\ &= \vec{v}\vec{u}^* + \vec{u}\vec{v}^* \\ &= A \end{aligned}$$

(b) Show that

$$A^2 = \vec{u}\vec{u}^* + \vec{v}\vec{v}^*.$$

$$\begin{aligned} A^2 &= (\vec{u}\vec{v}^* + \vec{v}\vec{u}^*)(\vec{u}\vec{v}^* + \vec{v}\vec{u}^*) \\ &= \vec{u}\cancel{\vec{v}^*\vec{u}}^{0}\vec{v}^* + \vec{u}\underbrace{\vec{v}^*\vec{v}}_{1}\vec{u}^* + \vec{v}\underbrace{\vec{u}^*\vec{u}}_{1}\vec{v}^* + \vec{v}\cancel{\vec{u}^*\vec{v}}^{0}\vec{u}^* \\ &= \vec{u}\vec{u}^* + \vec{v}\vec{v}^* \end{aligned}$$

(c) Show that $\vec{u} + \vec{v}$ is an eigenvector of A and find its corresponding eigenvalue.

$$\begin{aligned} A(\vec{u}+\vec{v}) &= (\vec{u}\vec{v}^* + \vec{v}\vec{u}^*)(\vec{u}+\vec{v}) \\ &= \vec{u}\cancel{\vec{v}^*\vec{u}}^{0} + \vec{u}\vec{v}^*\vec{v} + \vec{v}\vec{u}^*\vec{u} + \vec{v}\cancel{\vec{u}^*\vec{v}}^{0} \\ &= \vec{u} + \vec{v} \end{aligned}$$

Therefore, $\vec{u} + \vec{v}$ is an eigenvector with cooresponding eigenvalue $\lambda = 1$.