Partial Differential Equations
Spring 2024
Exam 1
02/16/24

Name (Print): Key

This exam contains 8 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

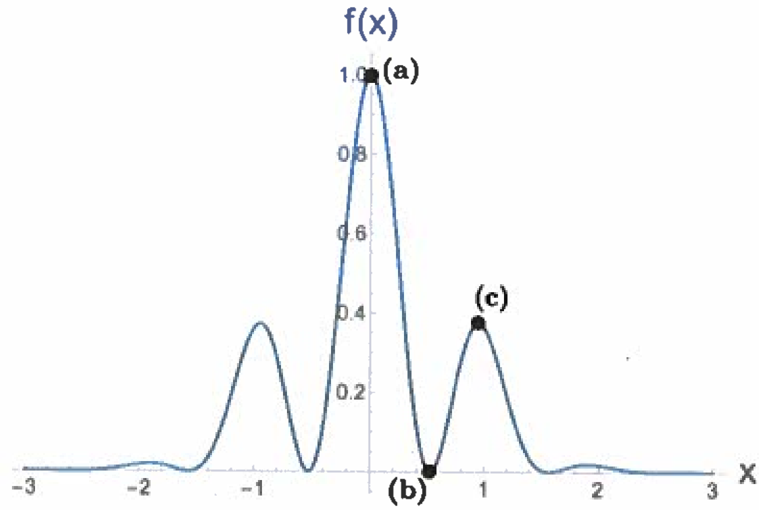
You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	15	
6	15	
7	15	
8	15	
Total:	100	

1. (10 points) In parts (a) and (b) below, $f(x)$ refers to the function plotted below.



(a) (5 points) Suppose $u(t,x)$ solves the following initial value problem:

$$u_t = u_{xx},\ x \in \mathbb{R},\ t > 0$$
$$u(0,x) = f(x).$$

Short Answer: At points **(a)**, **(b)**, and **(c)** is $u(t,x)$ increasing, decreasing or not changing in time at $t = 0$.

(a) decreasing
(b) increasing
(c) decreasing

(b) (5 points) Suppose $u(t,x)$ solves the following initial value problem:

$$u_{tt} = u_{xx},\ x \in \mathbb{R},\ t > 0$$
$$u(0,x) = f(x),$$
$$u_t(0,x) = f(x).$$

Short Answer: At points **(a)**, **(b)**, and **(c)** is $u(t,x)$ increasing, decreasing or not changing in time at $t = 0$?

(a) increasing
(b) not changing
(c) increasing

2. (10 points) **Short Answer:** Which of the following operators are linear? Circle any that are linear and cross out any that are nonlinear.

   (a) $L[u] = u_x + e^{-t}u_t$.

   (b) $L[u] = uu_x + u_t$.

   (c) $L[u] = u_x + (u_{tt})^2$.

   (d) $L[u] = u_x + u_t + \sin(x)$.

   (e) $L[u] = f(x)u_x + g(t)u_t$, where f and g are arbitrary smooth functions.

3. (10 points) Suppose $u(t, x)$ solves the following PDE

$$u_t = u_{xx} - u_x^2.$$

Show that $v(t, x) = e^{-u}$ solves the following PDE

$$v_t = v_{xx}.$$

$V_t = -U_t e^{-U} = (-U_{xx} + U_x^2) e^{-U}$

$V_x = -U_x e^{-U}$

$V_{xx} = -U_{xx} e^{-U} + U_x^2 e^{-U} = (-U_{xx} + U_x^2) e^{-U}$

$\Longrightarrow V_t = V_{xx}$

4. (10 points) **Short Answer:** For each of the below initial value problems defined for $x \in \mathbb{R}$ and $t > 0$, match them with the corresponding contour plot of the solution $u(t, x)$.

(*i*) (c)

$u_t = -u_x,$

$u(x, 0) = e^{-x^2}.$

(*ii*) (d)

$u_{tt} = u_{xx},$

$u(x, 0) = e^{-x^2},$

$u_t(x, 0) = 0.$

(*iii*) (a)

$u_{tt} = u_{xx},$

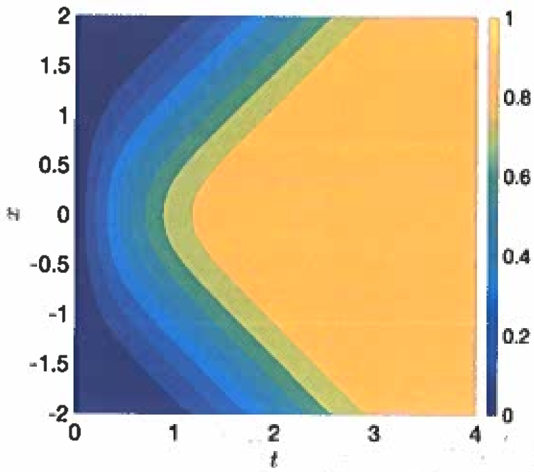
$u(x, 0) = 0,$

$u_t(x, 0) = e^{-x^2}.$

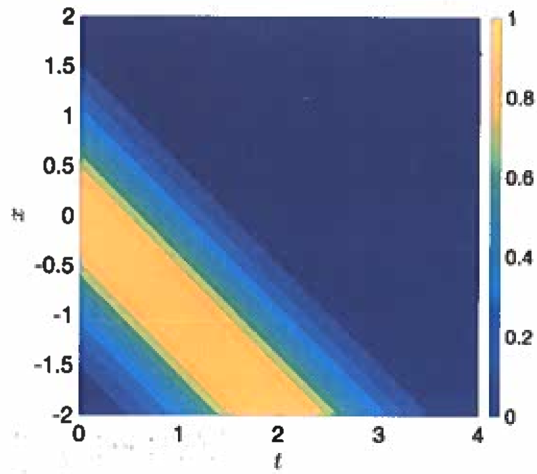
(*iv*) (b)

$u_t = u_x,$

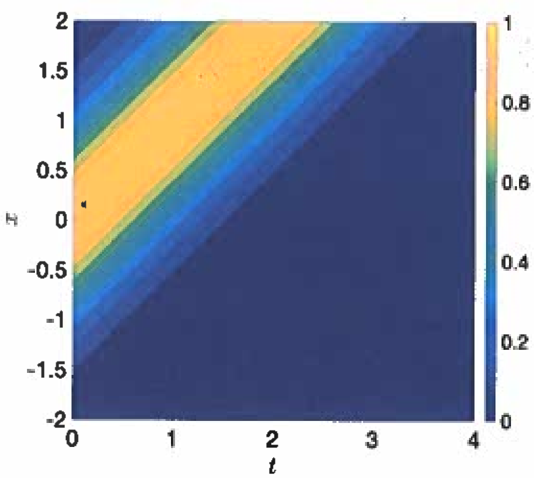
$u(x, 0) = e^{-x^2}.$



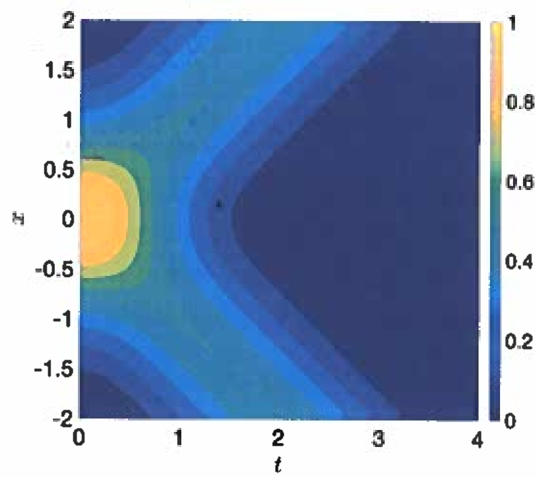
(*a*)



(*b*)



(*c*)

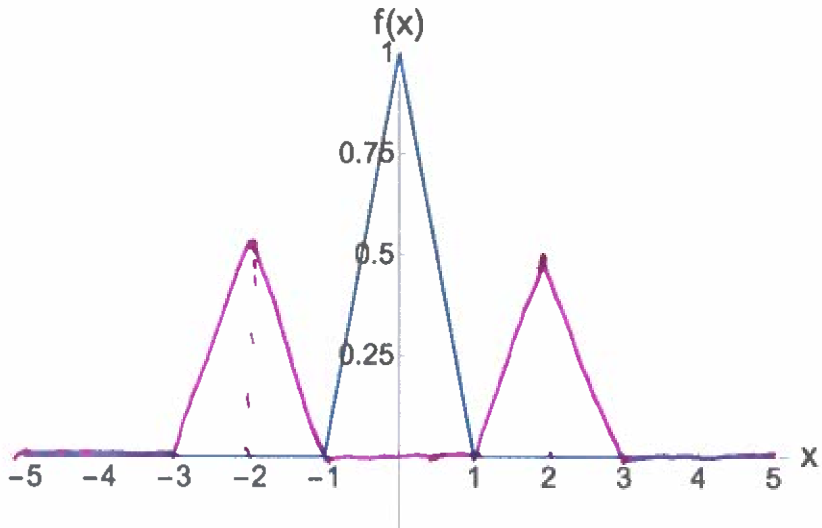


(*d*)

5. (15 points) Suppose $u(t,x)$ solves the following initial value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx},\ x \in \mathbb{R},\ t > 0 \\ u(0,x) &= f(x), \\ u_t(0,x) &= 0, \end{aligned}$$

where $f(x)$ is plotted below.



(a) (5 points) **Short Answer:** On the same set of axes, carefully sketch a graph of $u(1,x)$. Be sure to carefully sketch the location of any local maximum, minimum and zeros.

(b) (5 points) **Short Answer:** What is $\lim_{t\to\infty} u(t,x)$ for all $x \in \mathbb{R}$?

0

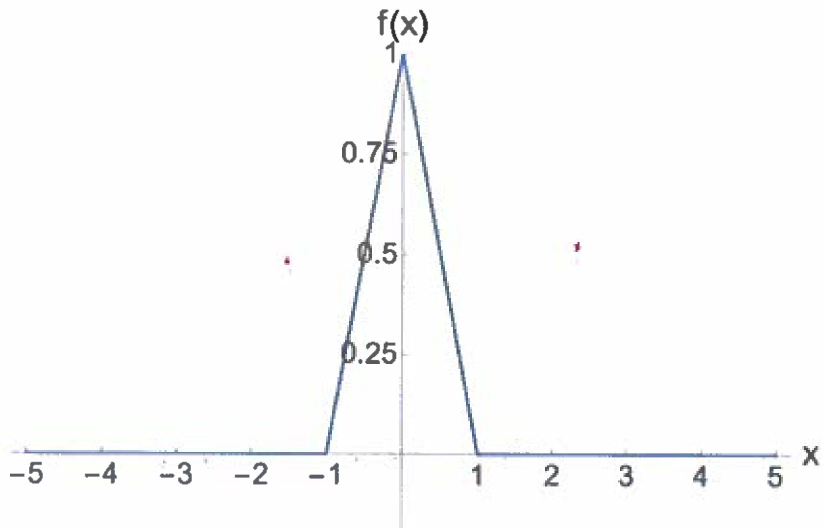
(c) (5 points) **Short Answer:** What is $\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)dx$?

1

6. (15 points) Suppose $u(t,x)$ solves the following initial value problem:

$$u_{tt} = 4u_{xx},\ x \in \mathbb{R},\ t > 0$$
$$u(0,x) = 0,$$
$$u_t(0,x) = f(x),$$

where $f(x)$ is plotted below.



(a) (5 points) **Short Answer:** Write down the generic form of the solution to this problem. **You do not have to evaluate the integrals. I am just looking for the formula.**

$$u(t,x) = \frac{1}{4}\int_{x-2t}^{x+2t} f(s)\,ds$$

(b) (5 points) **Short Answer:** What is $\lim_{t\to\infty} u(t,x)$ for all $x \in \mathbb{R}$?

$\frac{1}{4}$

(c) (5 points) **Short Answer:** What is $\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)dx$?

∞

7. (15 points) Consider the following PDE

$$u_{xx} = 0, \; x \geq 0, \; t > 0$$

subject to the boundary conditions:

$$u_x(t,0) = t^2,$$
$$u_t(t,0) = 1.$$

(a) (10 points) Find a solution this partial differential equation.

$U_x = f(t)$

$U = x f(t) + g(t)$

$\Rightarrow U_t = x f'(t) + g'(t)$

$\Rightarrow U_x(t,0) = f(t) = t^2$

$\Rightarrow U_t(t,0) = g'(t) = 1$

$\Rightarrow g(t) = t + c$

$\Rightarrow U(t,x) = x t^2 + t + c$

(b) (5 points) **Short Answer:** Are solutions to this equation unique?

No, since c can be any constant

8. (15 points) Solve the following initial value problem in the region $x \in \mathbb{R}$, $t > 0$:

$$u_t + \cos(t)u_x = -tu \text{ and } u(0, x) = \cos(x).$$

$$\frac{dx}{dt} = \cos(t)$$

$$x = \sin(t) + C$$

Let $z = x - \sin(t)$, $\tau = t$

$$\Rightarrow \frac{\partial u}{\partial \tau} = -\tau u$$

$$\Rightarrow \ln(u) = -\frac{\tau^2}{2} + f(z)$$

$$\Rightarrow u = f(z)e^{-\tau^2/2}$$

$$\Rightarrow u(t,x) = f(x - \sin(t))e^{-t^2/2}$$

$$\Rightarrow u(0,x) = f(x) = \cos(x)$$

$$\Rightarrow u(t,x) = \cos(x - \sin(t))e^{-t^2/2}$$