Partial Differential Equations
Spring 2024
Exam 2
03/29/24

Name (Print): Key

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

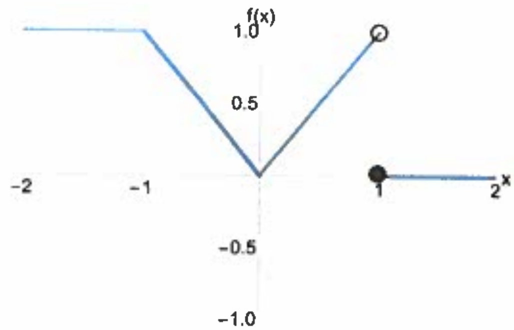
Problem	Points	Score
1	15	
2	10	
3	20	
4	15	
5	15	
6	10	
7	15	
Total:	100	

Do not write in the table to the right.

1. (15 points) The function $f(x)$ is defined on the domain $[-2, 2]$ by the formula

$$f(x) = \begin{cases} 1 & -2 \le x \le -1 \\ |x| & -1 < x < 1 \\ 0 & 1 \le x \le 2 \end{cases}$$

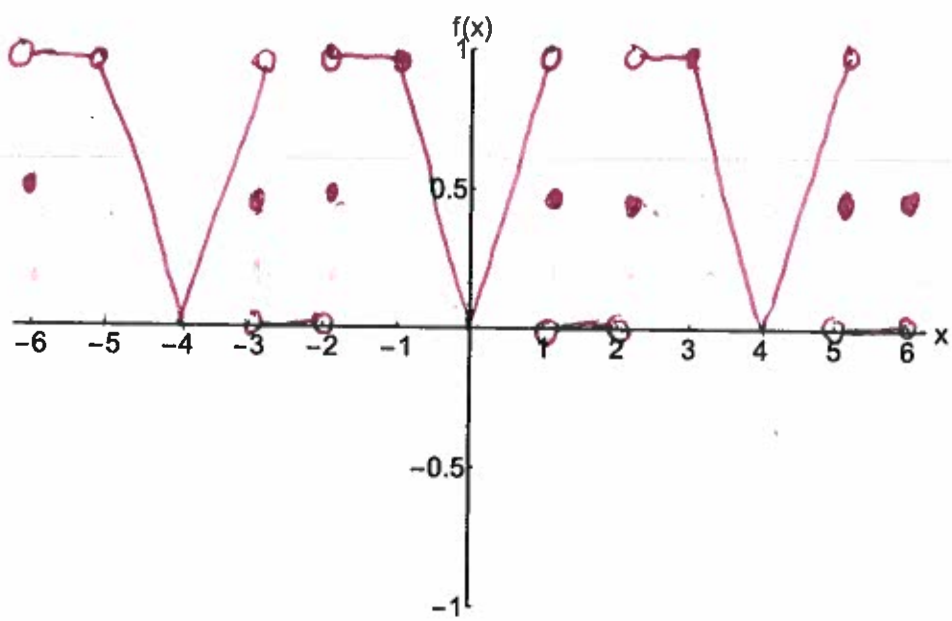
and is graphed below.



(a) (5 points) **Short Answer:** Given that $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ is an orthogonal system on $[-\pi, \pi]$, what is the corresponding orthogonal system on the interval $[-2, 2]$?

$\{1, \cos(\frac{\pi n x}{2}), \sin(\frac{\pi n x}{2})\}$, $n \in \mathbb{N}$.

(b) (10 points) On the set of axes below, sketch a graph of what the Fourier series for $f(x)$ converges to if you used the orthogonal system you found in part (a).



2. (10 points) **Short Answer:** The function $f(x)$ is plotted below. For each of the below initial boundary value problems defined for $x \in [0, \pi]$ and $t > 0$, match them with the corresponding contour plot of the solution $u(t, x)$.

(i) (B)

$$u_t = u_{xx},\quad u(t,0) = 0,\quad u(t,\pi) = 0,\quad u(0,x) = f(x).$$

(ii) (F)

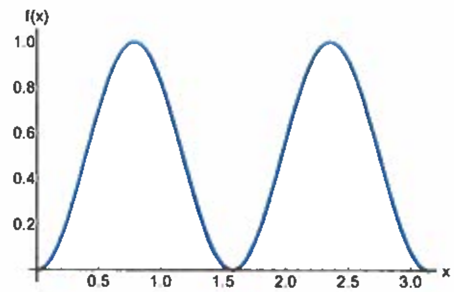
$$u_t = u_{xx},\quad u_x(t,0) = 0,\quad u_x(t,\pi) = 0,\quad u(0,x) = f(x).$$

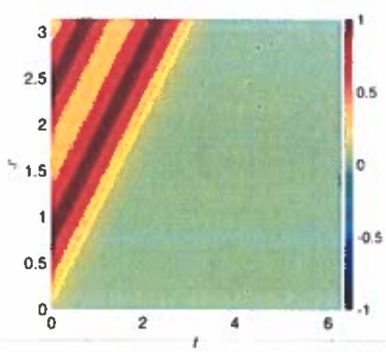
(iii) (C)

$$u_{tt} = u_{xx},\quad u(t,0) = 0,\quad u(t,\pi) = 0,\quad u(0,x) = f(x),\quad u_t(0,x) = 0.$$

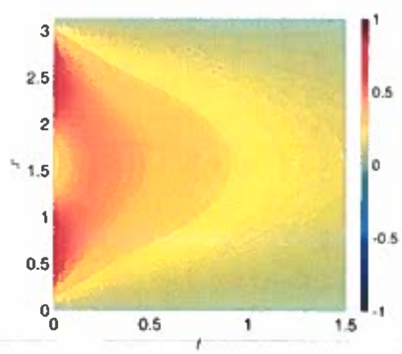
(iv) (E)

$$u_{tt} = u_{xx},\quad u_x(t,0) = 0,\quad u_x(t,\pi) = 0,\quad u(0,x) = f(x),\quad u_t(0,x) = 0.$$

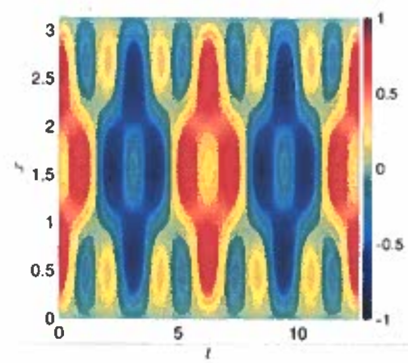




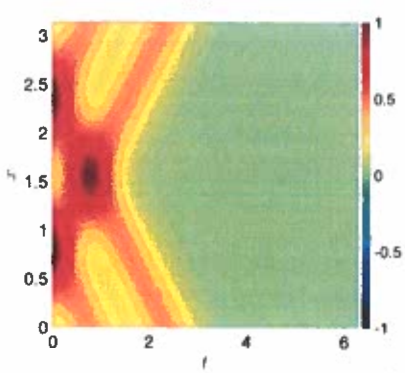
(a)



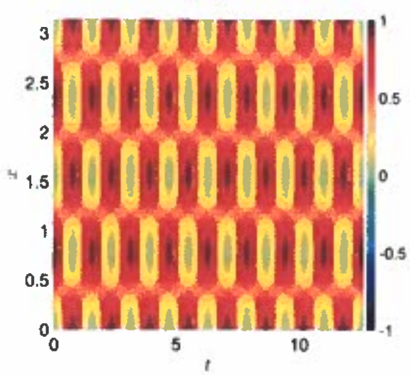
(b)



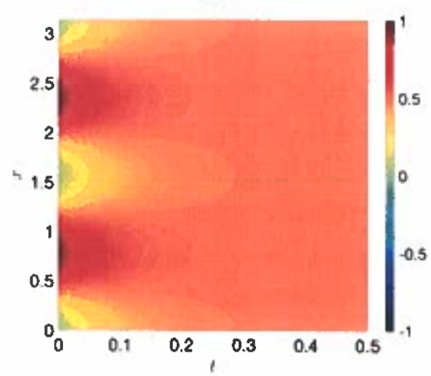
(c)



(d)



(e)



(f)

3. (20 points) The Fourier series for the function $f(x) = x^2$ on the interval $[-\pi, \pi]$ is given by

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

(a) (10 points) Using the Fourier series of $f(x)$, find the Fourier series of $g(x) = x$ on the interval $[-\pi, \pi]$.

$$2x \sim \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n} \sin(nx)$$

$$\Rightarrow x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

(b) (10 points) Using the Fourier series of $f(x)$ and $g(x)$, find the Fourier series of $h(x) = x^3$ on the interval $[-\pi, \pi]$.

$$\frac{x^3}{3} \sim \frac{\pi^2}{3} x + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3} \sin(nx)$$

$$\sim \frac{\pi^2}{3} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3} \sin(nx).$$

$$\Rightarrow x^3 \sim \pi^2 \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx).$$

4. (15 points) Find all separable solutions to the following initial boundary value problem

$$u_t = t^2 u_{xx}$$
$$u(t,0) = 0,$$
$$u(t,3) = 0.$$

Let $u = T \cdot X$

$\Rightarrow T' X = t^2 T X''$

$\Rightarrow \frac{1}{t^2}\frac{T'}{T} = \frac{X''}{X} = -w^2$

$\Rightarrow \frac{T'}{T} = -w^2 t^2, \quad X'' = -w^2 X$

$\Rightarrow \ln(T) = C - \frac{w^2 t^3}{3}, \quad X = A\cos(wx) + B\sin(wx)$

$\Rightarrow T = Ce^{-w^2 t^3/3}, \quad X = A\cos(wx) + B\sin(wx)$

Boundary conditions imply

$A = 0, \quad w = n\pi/3.$

$\Rightarrow u_n(t,x) = b_n e^{-n^2\pi^2 t^3/27} \sin(n\pi x/3)$

5. (15 points) Consider the following initial boundary value problem

$$\begin{aligned} u_t &= u_{xx} - u \\ u(0,x) &= f(x) \\ u_x(t,0) &= 0 \\ u_x(t,\pi) &= 0. \end{aligned}$$

Show that if $u(t,x)$ is a solution to this initial value problem then the energy defined by

$$E(t) = \int_0^\pi u(t,x)^2 dx$$

is decreasing in time.

$$\frac{dE}{dt} = \int_0^\pi 2 u_t u \, dx$$

$$= \int_0^\pi 2(u_{xx} - u) u \, dx$$

$$= \cancel{2 u_x u \Big|_0^\pi}^{0} - \int_0^\pi 2(u_x^2 + u^2) dx$$

$$= -2\int_0^\pi u_x^2 dx - 2\int_0^\pi u^2 dx$$

$$\leq 0.$$

6. (10 points) Consider the following initial boundary value problem:

$$u_t = u_{xx} + 9\sin(3x)$$
$$u(t,0) = u(t,\pi) = 0,$$
$$u(0,x) = \sin(x).$$

(a) (5 points) Write down a differential equation satisfied by a *steady state solution* to this differential equation.

$u_{xx} + 9\sin(3x) = 0$

(b) (5 points) Solve this equation for the steady state solution.

$\Rightarrow u_{xx} = -9\sin(3x)$

$\Rightarrow u = \sin(3x) + Ax + B$

Boundary conditions imply $A = B = 0$

$\Rightarrow u = \sin(3x)$.

7. (15 points) The following initial-boundary value problem models the vibrations of a string of length π:

$$u_{tt} = u_{xx},$$
$$u(t,0) = 0,$$
$$u(t,3) = 0,$$
$$u(0,x) = 3x - x^2$$
$$u_t(0,x) = 0.$$

Using separation of variables it can be shown that solution to this equation can be expressed as a Fourier series in the form

$$u(t,x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{3}t\right)\sin\left(\frac{n\pi}{3}x\right).$$

Using the initial conditions and orthogonality, find a formula for the coefficients b_n. You must analytically compute any integrals to receive full credit for this problem.

$$u(0,x) = 3x - x^2 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$\Rightarrow \int_0^3 (3x-x^2)\sin\left(\frac{n\pi x}{3}\right)dx = b_n \int_0^3 \sin^2\left(\frac{n\pi x}{3}\right)dx$$

$$= b_n \int_0^3 \frac{\left(1-\cos\left(\frac{2n\pi x}{3}\right)\right)}{2}dx$$

$$= \tfrac{3}{2} b_n$$

$$\Rightarrow b_n = \frac{2}{3}\int_0^3 (3x-x^2)\sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\underbrace{\cancel{(3x-x^2)\left(\frac{-3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right)\Big|_0^3}}_{0} + \int_0^3 (3-2x)\left(\frac{3}{n\pi}\right)\cos\left(\frac{n\pi x}{3}\right)dx\right.$$

$$= \frac{1}{n\pi}\left(\underbrace{\cancel{(3-2x)\cdot\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Big|_0^3}}_{0} + \frac{6}{n\pi}\int_0^3 2\sin\left(\frac{n\pi x}{3}\right)dx\right)$$

$$= \frac{6}{n^2\pi^2}\int_0^3 \sin\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{-18}{n^3\pi^3}\cos\left(\frac{n\pi x}{3}\right)\Big|_0^3$$

$$= \frac{-18}{n^3\pi^3}\left((-1)^n - 1\right)$$

$$= \frac{18}{n^3\pi^3}\left(1-(-1)^n\right).$$