MTH 352/652: Homework #10

Due Date: April 26, 2024

1 Problems for Everyone

1. pg. #283, #7.3.10, #7.3.11, #7.3.12, #7.3.14.
2. Find the convolution of the functions $f(x) = x$ and $g(x) = e^{-x^2}$.
3. Consider the following initial value problem for the heat equation with proportional heat loss:

$$\begin{aligned} u_t &= Du_{xx} - au, \ x \in \mathbb{R}, \ t > 0, \\ u(0,x) &= e^{-x^2}, \end{aligned}$$

   where $D > 0$ and $a > 0$ are constants. Using Fourier transforms find a formula for the solution to this initial value problem.

4. Consider the following initial value problem for the heat equation with advection:

$$\begin{aligned} u_t &= Du_{xx} - cu_x, \ x \in \mathbb{R}, \ t > 0, \\ u(0,x) &= e^{-x^2}, \end{aligned}$$

   where $D > 0$ and $c > 0$ are constants. Using Fourier transforms find a formula for the solution to this initial value problem.

5. Use Fourier transforms to find bounded solutions to the following differential equation on $\mathbb{R}$:

$$-u''(x) + u(x) = e^{-|x|}.$$

6. Consider the following initial value problem for the heat equation:

$$\begin{aligned} u_t &= Du_{xx}, \ x \in \mathbb{R}, \ t > 0, \\ u(0,x) &= f(x), \end{aligned}$$

   where $D > 0$ is a constant. Show that if $f(x)$ is an odd function then $u(t,x)$ is an odd function in x.

Homework #10: Solutions

#7.3.11

What is the convolution of a Gaussian Kernel e^{-x^2} with itself.

Solution:

$$\mathcal{F}[e^{-x^2} * e^{-x^2}] = \sqrt{2\pi}\, \mathcal{F}[e^{-x^2}]\, \mathcal{F}[e^{-x^2}]$$

$$= \frac{\sqrt{2\pi}}{2} e^{-k^2/4} \cdot e^{-k^2/4}$$

$$= \frac{\sqrt{2\pi}}{2} e^{-k^2/2}$$

Consequently,

$$e^{-x^2} * e^{-x^2} = \frac{\sqrt{2\pi}}{2} \mathcal{F}^{-1}[e^{-k^2/2}]$$

$$= \frac{\sqrt{2\pi}}{2} \cdot \sqrt{2}\, e^{-x^2/2}$$

$$= \sqrt{\pi}\, e^{-x^2/2}$$

$\blacksquare$

#2.

Find the convolution of the functions $f(x) = x$ and $g(x) = e^{-x^2}$.

Solution:

$$x * e^{-x^2} = \int_{-\infty}^{\infty} (x-y) e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} x e^{-y^2} dy - \cancelto{0}{\int_{-\infty}^{\infty} y e^{-y^2} dy}$$

$$= \sqrt{\pi}\, x$$

$\blacksquare$

#3

Consider the following initial value problem for the heat equation with proportional heat loss:

$$u_t = D u_{xx} - a u, \quad u(0,x) = e^{-x^2}$$

where $D>0$, $a>0$ are constants. Using Fourier transforms, find a formula for the solution to this initial value problem.

Solution:

$$u_t = D u_{xx} - a u$$

$$\Rightarrow \hat{u}_t = -k^2 D \hat{u} - a\hat{u}$$

$$\Rightarrow \hat{u} = \hat{u}(0,k) e^{-(k^2 D + a)t}$$

$$\Rightarrow u = \mathcal{F}^{-1}\left[\hat{u}(0,k) e^{-(k^2D+a)t}\right]$$

$$= e^{-at}\mathcal{F}^{-1}\left[\hat{u}(0,k) e^{-k^2 D t}\right]$$

$$= \frac{e^{-at}}{\sqrt{2\pi}} u(0,x) * \frac{1}{\sqrt{2Dt}} e^{-x^2/4Dt}$$

$$= \frac{e^{-at}}{\sqrt{4\pi D t}} e^{-x^2} * e^{-x^2/4Dt}.$$

#4

Consider the following initial value problem for the heat equation with advection

$$u_t = D u_{xx} - c u_x, \quad u(0,x) = e^{-x^2}$$

where $D; c > 0$. Using Fourier transforms, find a formula for the solution to this initial value problem.

Solution:

$$U_t = DU_{xx} - cU_x$$

$$\Rightarrow \hat{U}_t = -k^2 D\hat{U} - ick\hat{U}$$

$$\Rightarrow \hat{U} = \hat{U}(0,k)e^{-k^2Dt}e^{-ickt}$$

$$\Rightarrow U = \mathcal{F}^{-1}\left[\hat{U}(0,k)e^{-k^2Dt}e^{-ickt}\right]$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2} * \mathcal{F}^{-1}\left[e^{-k^2Dt}e^{-ickt}\right]$$

Now,

$$\mathcal{F}^{-1}\left[e^{-k^2Dt}e^{-ick}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2Dt}e^{-ickt}e^{ikx}\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k^2Dt}e^{ik(x-ct)}\,dk$$

$$= \mathcal{F}^{-1}\left[e^{-k^2Dt}\right](x-ct)$$

$$= \frac{1}{\sqrt{2Dt}} e^{-(x-ct)^2/4Dt}$$

Therefore,

$$u(t,x) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} e^{-(x-y)^2}e^{-(y-ct)^2/4Dt}\,dy.$$

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#5.

Use Fourier transforms to find bounded solutions to the following differential equation on $\mathbb{R}$:

$$-u''(x) + u(x) = e^{-|x|}.$$

Solution:

Taking Fourier transforms we have that

$$K^2\hat{U}+\hat{U}=\sqrt{\frac{2}{\pi}}\frac{1}{1+K^2}$$

$$\Rightarrow \hat{U}=\sqrt{\frac{2}{\pi}}\frac{1}{(1+K^2)^2}$$

$$\Rightarrow U=\sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2}e^{-|x|}*e^{-|x|}$$

$$=\sqrt{\frac{\pi}{2}}e^{-|x|}*e^{-|x|}.$$

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#6.

Consider the following initial value problem for the heat equation:

$$U_t=DU_{xx},\quad x\in\mathbb{R},\ t>0$$

$$U(0,x)=f(x).$$

Show that if $f(x)$ is an odd function then $U(t,x)$ is an odd function in x.

Solution:

The solution to this problem is given by

$$U(t,x)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}f(x-y)e^{-y^2/4Dt}\,dy$$

Therefore,

$$U(t,-x)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}f(-x-y)\exp\left(\frac{-y^2}{4Dt}\right)dy$$

$$=\frac{-1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}f(x+y)\exp\left(\frac{-y^2}{4Dt}\right)dy,\quad \text{let } u=-y,\ du=-dy$$

$$=\frac{1}{\sqrt{4\pi Dt}}\int_{\infty}^{-\infty}f(x-u)\exp\left(\frac{-u^2}{4Dt}\right)du$$

$$=-U(t,x).$$

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