MTH 352/652: Homework #5

Due Date: March 01, 2024

1 Problems for Everyone

1. pg. 81, #3.2.14-3.2.16, #3.2.20
2. pg. 87-88, #3.2.34-#3.2.~~25~~ 35
3. pg. 95, #3.3.1-3.3.3
4. pg. 97, #3.4.3
5. Find the Fourier series of the function $|\sin(x)|$ in the interval $(-\pi, \pi)$. Use it to find the following sums:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

6. Let $\phi(x) = x$.
   (a) Find the Fourier series of $\phi(x)$ on the interval $(0, L)$, where $L > 0$ is a constant.
   (b) Integrating term by term, find the Fourier series of $x^2/2$ on the interval $(0, L)$. Do not forget about the a_0 term.
   (c) Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

   (d) Find the Fourier series of x^3 and x^4 on the interval $(0, L)$.
   (e) Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

Homework #5

pg. 81. #3.2.15-3.2.16

Graph the folling piecewise continuous functions. List all discontinities, jump magnitudes, determine if the functions are piecewise C^1, list all corners.

(a) $\begin{cases} e^x, & 1<|x|<2 \\ 0, & \text{o.w.} \end{cases}$

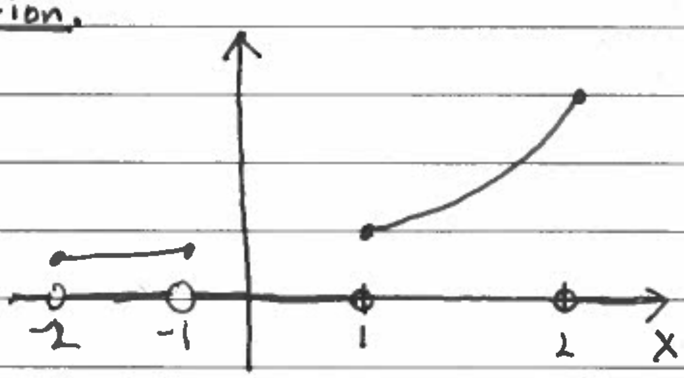
(c) $\begin{cases} \frac{\sin(x)}{x}, & 0<|x|<2\pi \\ 1, & x=0 \\ 0, & \text{o.w.} \end{cases}$

(d) $\begin{cases} x, & |x|\leq 1 \\ x^2, & |x|>1 \end{cases}$

(e) $\begin{cases} x, & -1<x<0 \\ \sin(x), & 0<x<\pi \\ 0, & \text{o.w.} \end{cases}$

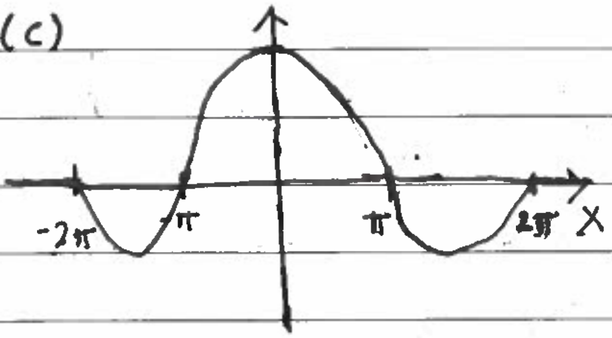
Solution:

(a)



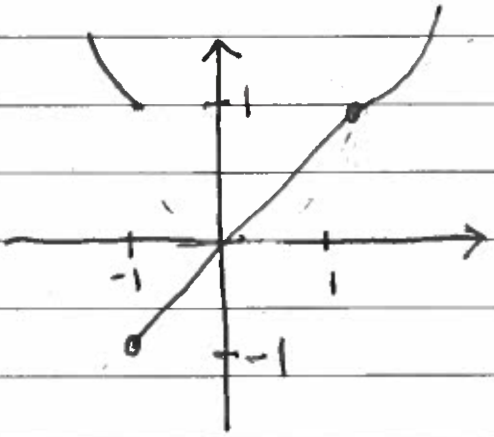
Discontinuous at $x=\pm1,\pm2$ with a jumps of $e^{-2}, e^{-1}, e^{1}, e^{2}$ at $x=-2,-1,1,2$ respectively. The function is piecewise C^1 with no corners.

(c)



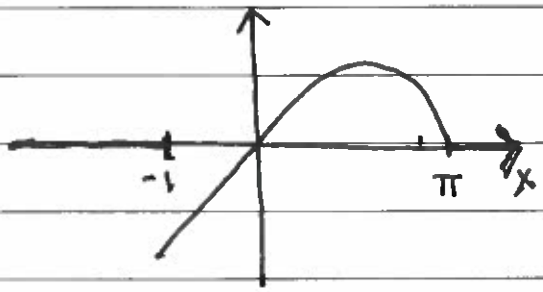
f(x) is continous and is piecewise C^1 with corners at $x=-2\pi$ and $x=2\pi$.

(d)



f(x) has a jump discontinuity at $x=-1$ of magnitude 2 and is piecewise C^1 with a corner at $x=1$.

e)



f(x) has a jump discontinuity at $x=-1$ of magnitude -1 and is piecewise C^1 with no corners.

■

pg. 95, #3.3.1

Starting with the Fourier series for the step function $\sigma(x)$, use integration to:

(a) Find the Fourier series for the ramp function

$$\rho(x) = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(b) Then, find the Fourier series for the second order ramp function

$$\rho_2(x) = \begin{cases} \frac{1}{2}x^2, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Solution:

Recall that

$$\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n\ge1}^{\infty} \frac{1}{2n-1}\sin((2n-1)x)$$

$$x \sim 2\sum_{n\ge1}^{\infty} (-1)^{n+1}\frac{1}{n}\sin(nx)$$

(a) Antidiferentiating, we have

$$\begin{aligned}\rho(x) &= \int \sigma(x)\,dx \\ &= c + \tfrac{1}{2}x - \tfrac{2}{\pi}\sum_{n\ge1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x) \\ &= c + \sum_{n\ge1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin(nx) - \frac{2}{\pi}\sum_{n\ge1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x)\end{aligned}$$

By orthogonality, we have that

$$\int_{-\pi}^{\pi}\rho(x)\,dx = \int_{-\pi}^{\pi} c\,dx$$

$$\Rightarrow \int_0^{\pi} x\,dx = 2\pi c$$

$$\Rightarrow \frac{1}{2}\pi^2 = 2\pi c$$

$$\Rightarrow c = \frac{1}{4}\pi$$

Therefore,

$$\rho(x) = \frac{1}{4}\pi + \sum_{n\ge1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin(nx) - \frac{2}{\pi}\sum_{n\ge1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x).$$

Antidifferentiating again we have

$$g_2(x) = c + \frac{\pi}{4}x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)x)$$

$$= c + \frac{\pi}{2}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}\sin(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)x)$$

$$\Rightarrow \int_{-\pi}^{\pi} g_2(x)\,dx = 2\pi c$$

$$\Rightarrow \int_0^{\pi} \tfrac{1}{2}x^2\,dx = 2\pi c$$

$$\Rightarrow c = \frac{\pi^3}{12}$$

Therefore,

$$g_2(x) = \frac{\pi^3}{12} + \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\sin((2n-1)x).$$

■

#4. pg. 97, #3.4.3

Find the Fourier series for the following functions on the indicated interval and graph the function that the Fourier series converges to.

(a) $|x|$, $-3 \le x \le 3$

(b) $x^2 - 4$, $-2 \le x \le 2$

(d) $\sin(x)$, $-1 \le x \le 1$

(e) $\sigma(x)$, $-2 \le x \le 2$

Solution:

(a) Since $|x|$ is even it follows that

$$|x| \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{3}\right)$$

$$\Rightarrow \int_{-3}^{3} |x|\,dx = 2\int_0^3 x\,dx = 3a_0$$

$$\Rightarrow a_0 = \tfrac{1}{3}\cdot 9 = 3$$

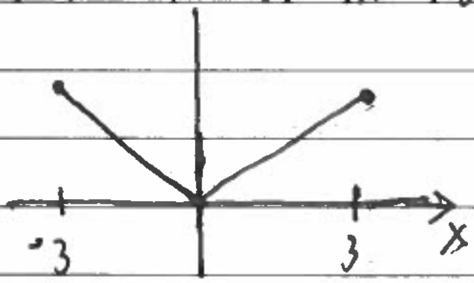
$$\Rightarrow \int_{-3}^{3} \cos(\pi n x/3)\,|x|\,dx = \int_{-3}^{3} a_n \cos^2(n\pi x/3)\,dx$$
$$= 3a_n$$

$$\Rightarrow \frac{2}{3}\int_0^3 \cos(\pi n x/3)\,x\,dx = a_n$$

$$\Rightarrow a_n = \frac{2}{3}\left(x \cdot \frac{3}{\pi n} \sin\left(\frac{\pi n x}{3}\right)\Big|_0^3 - \frac{3}{\pi n}\int_0^3 \sin\left(\frac{\pi n x}{3}\right)dx \right)$$
$$= \frac{2}{3}\cdot\frac{3}{\pi n}\cdot\frac{3}{\pi n}\cos\left(\frac{\pi n x}{3}\right)\Big|_0^3$$
$$= \frac{6}{\pi^2 n^2}\left((-1)^{n+1} - 1\right)$$

$$\Rightarrow |x| \sim \frac{3}{2} - \frac{12}{\pi^2}\sum_{n\geq 1}^{\infty} \frac{1}{4n^2}\cos(2\pi n x/3).$$

Since $\tilde{f}(x)$ is continuous the Fourier series converges to



(b) Since x^2-4 is even it follows that

$$x^2 - 4 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

We can just find the Fourier series for x^2 and add 4!

$$x^2 \sim \frac{a_0'}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow \int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = \int_{-2}^{2} \frac{a_0'}{2}\,dx$$

$$\Rightarrow \frac{2}{3}\cdot 8 = 2a_0'$$

$$\Rightarrow a_0' = \frac{8}{3}.$$

$$\Rightarrow \int_{-2}^{2} x^2 \cos\left(\frac{n\pi x}{2}\right) = \int_{-2}^{2} a_n \cos^2\left(\frac{n\pi x}{2}\right) dx = 2a_n$$

$$\Rightarrow a_n = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2}{n\pi}\left(\cancel{x^2 \sin\left(\frac{n\pi x}{2}\right)}^{0}\Big|_0^2 - \int_0^2 2x \sin\left(\frac{n\pi x}{2}\right) dx \right)$$

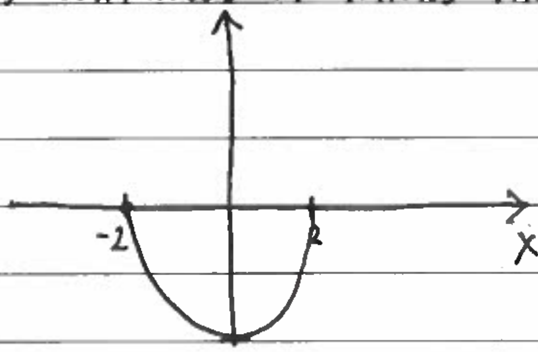
$$= \frac{2^2}{n^2\pi^2}\left(2x\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \cancel{\int_0^2 2\cos\left(\frac{n\pi x}{2}\right) dx}^{0} \right)$$

$$= \frac{16}{n^2\pi^2}(-1)^n$$

$$\Rightarrow x^2 - 4 \sim \frac{8}{6} - 4 + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow x^2 - 4 \sim -\frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right).$$

Since $F(x)$ is continous it follows that the Fourier series converges to



(d) Since $\sin(x)$ is odd it follows that

$$\sin(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\Rightarrow \int_{-1}^{1} \sin(x)\sin(n\pi x)\,dx = 2b_n$$

$$\Rightarrow b_n = \frac{1}{2}\int_{-1}^{1} \sin(x)\sin(n\pi x)\,dx$$

$$\Rightarrow b_n = \frac{1}{2}\int_{-1}^{1} \frac{(e^{ix}-e^{-ix})}{2i}\cdot\frac{(e^{n\pi i x}-e^{-n\pi i x})}{2i}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{e^{i(1+n\pi)x}-e^{i(1-n\pi)x}-e^{-i(1-n\pi)x}+e^{-i(1+n\pi)x}}{-4}\,dx$$

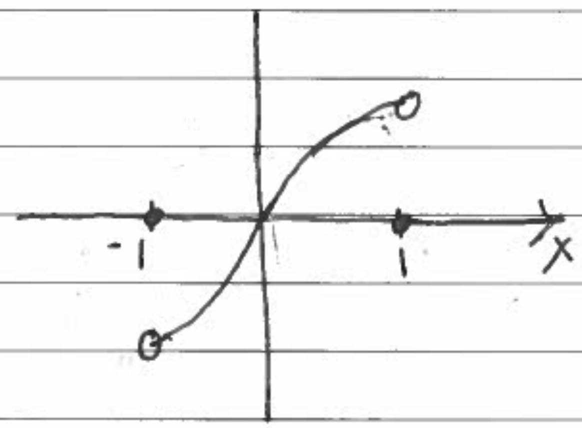
$$= -\frac{1}{2}\int_{-1}^{1}\left(\cos((1+n\pi)x)-\cos((1-n\pi)x)\right)dx$$

$$= -1\int_{0}^{1}\left(\cos((1+n\pi)x)-\cos((1-n\pi)x)\right)dx$$

$$= \frac{\sin((1-n\pi))}{1-n\pi} - \frac{\sin((1+n\pi))}{1+n\pi}$$

$$\Rightarrow \sin(x) \sim \sum_{n=1}^{\infty}\left(\frac{\sin(1-n\pi)}{1-n\pi} - \frac{\sin(1+n\pi)}{1+n\pi}\right)\sin(nx)$$

Since $\tilde{f}(x)$ is not continuous, the Fourier series converges to the following function.



(e) The Fourier series on this domain is given by

$$\sigma(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$\Rightarrow \int_{-2}^{2}\sigma(x)dx = \int_{-2}^{2}\frac{a_0}{2}dx = 2a_0$$

$$\Rightarrow \int_{0}^{2} x\,dx = 2a_0$$

$$\Rightarrow 2 = a_0$$

$$\Rightarrow a_0 = 1.$$

We also have that

$$\int_{-2}^{2}\sigma(x)\cos\left(\frac{\pi n x}{2}\right)dx = \int_{-2}^{2} a_n\cos^2\left(\frac{\pi n x}{2}\right)dx$$

$$= \int_{-2}^{2} a_n\left(\frac{1+\cos(\pi n x)}{2}\right)dx$$

$$= 2a_n$$

$$\Rightarrow \int_{0}^{2} x\cos\left(\frac{\pi n x}{2}\right)dx = 2a_n$$

$$\Rightarrow \frac{2}{\pi n}\cancelto{0}{x\sin\left(\frac{\pi n x}{2}\right)}\Bigg|_{0}^{2} - \frac{2}{\pi n}\int_{0}^{2}\sin\left(\frac{\pi n x}{2}\right)dx = 2a_n$$

$$\Rightarrow \frac{4}{\pi^2 n^2}\cos(\pi n) = 2a_n$$

$$\Rightarrow a_n = \frac{2}{\pi^2 n^2}(-1)^n$$

Furthermore,

$$\int_{-2}^{2}\sigma(x)\sin\left(\frac{\pi n x}{2}\right)dx = \int_{-2}^{2} b_n\sin^2\left(\frac{\pi n x}{2}\right)dx$$

$$\Rightarrow \int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx = 2b_n$$

$$\Rightarrow -\frac{2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\Bigg|_{0}^{2} + \int_{0}^{2}\frac{2}{n\pi}\cancelto{0}{\cos\left(\frac{n\pi x}{2}\right)}dx = 2b_n$$

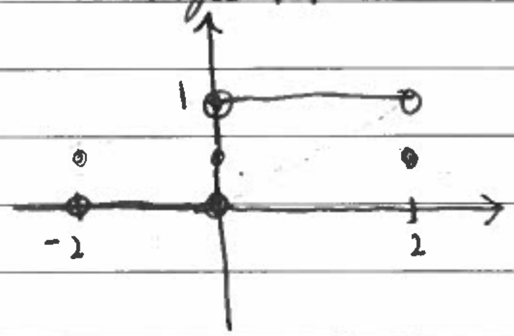
$$\Rightarrow \frac{4}{n\pi}(-1)^{n+1} = 2b_n$$

$$\Rightarrow b_n = \frac{2}{n\pi}(-1)^{n+1}$$

Therefore,

$$\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$

Since $\tilde{\sigma}(x)$ is discontinuous at $x=-2$ and $x=2$ it follows that the Fourier series converges to.



#5

Find the Fourier series of the function $|\sin(x)|$ on the interval $(-\pi,\pi)$. Use it to find the following sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Solution:

Since $|\sin(x)|$ is even function it follows that

$$|\sin(x)| \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} |\sin(x)|dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx$$

$$\Rightarrow 2\int_0^{\pi} \sin(x)dx = \pi a_0$$

$$\Rightarrow -2\cos(x)\Big|_0^{\pi} = \pi a_0$$

$$\Rightarrow a_0 = 4/\pi$$

We also have that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\cos(nx)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{(e^{ix}-e^{-ix})}{2i}\frac{(e^{inx}+e^{-inx})}{2}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{e^{i(1+n)x}+e^{i(1-n)x}-e^{-i(1-n)x}-e^{-i(1+n)x}}{4i}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin((1-n)x)+\sin((1+n)x))\,dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos((1-n)x)}{1-n}\Big|_0^{\pi} + \frac{\cos((1+n)x)}{1+n}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-\frac{(-1)^{n-1}}{1-n} + \frac{1}{1-n} - \frac{(-1)^{n+1}}{1+n} + \frac{1}{1+n}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n}{1-n} + \frac{1}{1-n} + \frac{(-1)^n}{1+n} + \frac{1}{1+n}\right)$$

$$\Rightarrow a_n = \frac{1}{\pi}\left(\frac{(-1)^n+1}{1-n^2}\right)$$

Therefore,

$$|\sin(x)| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{1-4n^2}$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2-1}$$

Consequently, substituting we have

1. $\underline{x=0}$: $0 = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = 1.$$

2. $\underline{x=\pi/2}$: $1 = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$

$$= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi}{2} - 1 = \frac{\pi-2}{2}.$$

#6.

Let $\phi(x) = x$.

(a) Find the Fourier series of $\phi(x)$ on $(0, L)$ where $L > 0$ is a constant.

(b) Find the Fourier series of $x^2/2$ on $(0, L)$.

(c) Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

(d) Find the Fourier series of x^3 and x^4 on $(0, L)$.

(e) Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^4}.$$

Solution:

(a) The Fourier series for $\phi(x)$ satisfies

$$x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right)$$

$$\Rightarrow \int_0^L x\,dx = \int_0^L \frac{a_0}{2}\,dx$$

$$\Rightarrow L^2/2 = a_0 L/2$$

$$\Rightarrow a_0 = L$$

$$\Rightarrow \int_0^L x \cos\left(\frac{2n\pi x}{L}\right) dx = \int_0^L a_n \cos^2\left(\frac{2n\pi x}{L}\right) dx$$

$$\Rightarrow \frac{L}{2n\pi} x \cancelto{0}{\sin\left(\frac{2n\pi x}{L}\right)}\Bigg|_0^L - \frac{L}{2n\pi}\int_0^L \cancelto{0}{\sin\left(\frac{2n\pi x}{L}\right)} dx = \frac{a_n L}{2}$$

$$\Rightarrow a_n = 0$$

$$\Rightarrow \int_0^L x \sin\left(\frac{2n\pi x}{L}\right) dx = \frac{b_n \cdot L}{2}$$

$$\Rightarrow -\frac{L}{2n\pi} x \cos\left(\frac{2n\pi x}{L}\right)\Bigg|_0^L + \frac{L}{n\pi}\int_0^L \cancelto{0}{\cos\left(\frac{2n\pi x}{L}\right) dx} = \frac{b_n L}{2}$$

$$-\frac{L^2}{2n\pi} = \frac{b_n L}{2}$$

$$\Rightarrow b_n = -\frac{L}{n\pi}.$$

Therefore,

$$x \sim \frac{L}{2} - \frac{L}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{L}\right)$$

(b) Integrating we have that

$$\frac{x^2}{2} \sim C + \frac{Lx}{2} + \frac{L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{2n^2} \cos\left(\frac{2n\pi x}{L}\right)$$

$$\sim c + \frac{L^2}{4} - \frac{L^2}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{L}\right) + \frac{L^2}{2\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{L}\right)$$

$$\sim C - \frac{L^2}{2\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{L}\right) + \frac{L^2}{2\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{L}\right)$$

$$\Rightarrow \int_0^L \frac{x^2}{2} dx = \int_0^L c\, dx$$

$$\Rightarrow \frac{L^3}{6} = Lc$$

$$\Rightarrow c = L^2/6$$

Therefore,

$$\frac{x^2}{2} \sim \frac{L^2}{6} - \frac{L^2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi x}{L}\right) + \frac{L^2}{2\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{L}\right)$$

(c) Substituting in $x = L/2$ we obtain

$$\frac{L^2}{8} - \frac{L^2}{6} = \frac{L^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 2\pi^2\cdot\left(\frac{4-3}{24}\right) = \frac{\pi^2}{12}$$

(d) Integrating again we have that

$$\frac{x^3}{6} \sim c + \frac{L^2}{6}x + \frac{L^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right)$$

$$\Rightarrow x^3 \sim c + L^2 x + \frac{L^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right)$$

$$\sim c - \frac{L^3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(2\right.$$

$$\Rightarrow \int_0^L x^3\,dx = cL$$

$$\Rightarrow c = L^3/4$$

$$\Rightarrow x^3 \sim \frac{L^3}{4} - \frac{L^3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^3}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right)$$

Integrating again we have that

$$\frac{x^4}{4} \sim c + \frac{L^3}{4}x + \frac{L^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^4}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right) - \frac{L^4}{8\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\cos\left(\frac{2n\pi}{L}\right.$$

$$\Rightarrow x^4 \sim c - \frac{L^4}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{L}\right) + \frac{L^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^4}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right) - \frac{L^4}{8\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\cos\left(\frac{2n\pi}{L}\right.$$

Integrating, we have that

$$\frac{L^5}{5} = cL \Rightarrow c = \frac{L^4}{5}$$

$$\Rightarrow x^4 \sim \frac{L^4}{5} - \frac{L^4}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{L}\right) + \frac{L^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{L}\right) + \frac{L^4}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{2n\pi x}{L}\right) - \frac{L^4}{8\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\mathrm{co}$$

Substituting in $x = L/2$ we have that

$$\frac{L^4}{16} = \frac{L^4}{5} + \frac{L^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} - \frac{L^4}{8\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\Rightarrow \frac{L^4}{16} = \frac{L^4}{5} - \frac{L^4}{24} - \frac{L^4}{8\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = 8\pi^4\left(\frac{1}{5} - \frac{1}{24} - \frac{1}{16}\right)$$

$$= 8\pi^4\left(\frac{23}{240}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{23\pi^4}{30}$$

$\blacksquare$