MTH 352/652: Homework #9

Due Date: April 19, 2024

1 Problems for Everyone

1. pg. 273-274 #7.1.1-7.1.4, #7.1.15, 7.1.16.
2. Compute $\mathcal{F}\left[xe^{-ax^2}\right](k)$, where $a > 0$ is a constant.
3. Given that

$$\mathcal{F}\left[xe^{-|x|}\right](k) = -\frac{4ik}{(1+k^2)^2},$$

find

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right].$$

Homework #9

pg. 273, #7.1.1

Find the Fourier transforms of the following functions

(a) $e^{-(x+4)^2}$

(c) $\begin{cases} x, & |x|\le 1 \\ 0, & \text{o.w.} \end{cases}$

(d) $\begin{cases} e^{-2x}, & x\ge 0 \\ e^{3x}, & x\le 0 \end{cases}$

(f) $\begin{cases} e^{-x}\sin(x), & x\ge 0 \\ 0, & x\le 0. \end{cases}$

Solution:

(a)
$$\mathcal{F}[e^{-(x+4)^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+4)^2} e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{-ik(u-4)}\,dx$$
$$= e^{4ik}\,\mathcal{F}[e^{-u^2}]$$
$$= e^{4ik}\frac{1}{\sqrt{2}}e^{-k^2/4}.$$

(c)
$$\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x(\cos(kx) - i\sin(kx))\,dx$$
$$= \frac{-i2}{\sqrt{2\pi}}\int_{0}^{1} x\sin(kx)\,dx$$
$$= -i\sqrt{\frac{2}{\pi}}\left(-\frac{x\cos(kx)}{k}\Big|_0^1 + \int_0^1 \frac{\cos(kx)}{k}\,dx\right)$$
$$= i\sqrt{\frac{2}{\pi}}\frac{\cos(k)}{k} - i\sqrt{\frac{2}{\pi}}\frac{\sin(k)}{k^2}$$

(d) $\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{3x} e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2x} e^{-ikx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(3-ik)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(2+ik)} dx$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{3-ik} + \frac{1}{\sqrt{2\pi}} \frac{1}{2+ik}$

(f) $\mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} \sin(x) e^{-ikx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-x}(e^{ix} - e^{-ix}) e^{-ikx}}{2i} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-i+ik)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1+i+ik)x} dx$

$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i+ik} + \frac{1}{1+i+ik} \right)$

$= \sqrt{\frac{2}{\pi}} \frac{(1+i)}{1+(k-1)^2}.$

■

<u>pg. 274, # 7.1.15</u>

Given that the Fourier transform of $f(x)$ is $\hat{f}(k)$, find the Fourier transform of $g(x) = f(ax+b)$, where $a, b \in \mathbb{R}$.

<u>Solution:</u>

$\mathcal{F}[g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(ax+b) dx$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{a} \int_{-\infty}^{\infty} e^{-ik\left(\frac{u-b}{a}\right)} f(u) du$

$= \frac{e^{ikb/a}}{a} \hat{f}\left(\frac{k}{a}\right).$

■

~~pg.~~ 274, #7.1.16

Let a be a real constant. Given the Fourier transform $\hat{f}(k)$ of $f(x)$, find the Fourier transform of (a) $f(x)e^{iax}$, (b) $f(x)\cos(ax)$, (c) $f(x)\sin(ax)$.

Solution:

(a) $$\mathcal{F}[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}e^{-ikx}f(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(k-a)x}f(x)\,dx$$
$$= \hat{f}(k-a).$$

(b) $$\mathcal{F}[\cos(ax)f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{iax}+e^{-iax}}{2}\, e^{-ikx}f(x)\,dx$$
$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-i(k-a)x}+e^{-i(k+a)}\right)f(x)\,dx$$
$$= \frac{1}{2}\left(\hat{f}(k-a)+\hat{f}(k+a)\right)$$

(c) $$\mathcal{F}[\sin(ax)f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{iax}-e^{-iax}}{2i}\, e^{-ikx}f(x)\,dx$$
$$= \frac{1}{2i}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-i(k-a)x}-e^{-i(k+a)x}\right)f(x)\,dx$$
$$= \frac{1}{2i}\left(\hat{f}(k-a)-\hat{f}(k+a)\right)$$

#2.

Compute $\mathcal{F}[xe^{-ax^2}]$, where $a>0$ is a constant.

Solution:

$$\mathcal{F}[xe^{-ax^2}] = \mathcal{F}\left[-\frac{1}{2a}\frac{d}{dx}(e^{-ax^2})\right]$$
$$= -\frac{1}{2a}(ik)\,\mathcal{F}[e^{-ax^2}]$$
$$= \frac{-ik}{2a}\frac{1}{\sqrt{2a}}e^{-k^2/4a}.$$
$$= \frac{-ik}{(2a)^{3/2}}e^{-k^2/4a}$$

#3.

Given that

$$\mathcal{F}[xe^{-|x|}] = -\frac{4ik}{(1+k^2)^2}$$

find

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right].$$

Solution:

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{(1+x^2)^2}e^{-ikx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-u}{(1+u^2)^2}e^{iku}\,du$$
$$= \frac{1}{4i}ke^{-|k|}$$
$$= -\frac{1}{4}ike^{-|k|}$$