MTH 352/652: Homework #2

Due Date: February 02, 2024

1 Problems for Everyone

1. Let $c \neq 0$ and suppose $u(t,x)$ solves the following initial value problem

$$u_t + cu_x = 0 \text{ and } u(0,x) = f(x).$$

   Suppose f is continuous and satisfies $\lim_{|x|\to\infty} f(x) = 0$. Prove that $\lim_{t\to\infty} u(t,x) = 0$.

2. Solve the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$
   (a) $u_t + xtu_x = 0$ and $u(0,x) = f(x)$
   (b) $u_t + xu_x = e^t$ and $u(0,x) = f(x)$

3. Solve the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$
   (a) $u_t + xu_x = -tu$ and $u(0,x) = f(x)$
   (b) $tu_t + xu_x = -2u$ and $u(0,x) = f(x)$
   (c) $u_t + u_x = -tu$ and $u(0,x) = f(x)$

4. Consider the following initial value problems in the region $x \in \mathbb{R}$, $t > 0$:

$$u_t + u_x + u^2 = 0 \text{ and } u(0,x) = f(x).$$

   (a) Find the general solution to this initial value problem.
   (b) Show that if $f(x)$ is bounded and positive, i.e., $0 \leq f(x) \leq M$, then the solution exists for all $t > 0$ and

$$\lim_{t\to\infty} u(t,x) = 0.$$

   (c) Show that if $f(x)$ is negative, so $f(x) < 0$ at some $x \in \mathbb{R}$, then the solution blows up in finite time:

$$\lim_{t\to\tau^-} u(t,y) = -\infty$$

   for some $\tau > 0$ and some $y \in \mathbb{R}$.

5. Consider the equation

$$u_t + xu_x = 0$$

   with the boundary condition $u(t,0) = \phi(t)$.
   (a) For $\phi(t) = t$, show that no solution exists.
   (b) For $\phi(t) = 1$, show that there are infinitely many solutions.