MTH 352/652: Homework #4

Due Date: February 23, 2024

1 Problems for Everyone

1. pg. 72, #3.12, #3.14, #3.15, #3.16.

2. Consider the following initial boundary value problem on the domain $[0, \pi]$:

$$\begin{cases} tu_t = u_{xx} + 2u \\ u(t,0) = u(t,\pi) \\ u(0,x) = 0 \end{cases}.$$

   By separating variables, show that this initial boundary value problem has an infinite number of solutions.

3. By considering the energy

$$E(t) = \int_{-\pi}^{\pi} u^2 dx,$$

   prove that solutions to the heat equation

$$\begin{cases} u_t = u_{xx} \\ u(0,x) = f(x) \\ u(t,-\pi) = u(t,\pi) \\ u_x(t,-\pi) = u_x(t,\pi) \end{cases}$$

   are unique. **Hint:** First prove that $E(t)$ is decreasing in time.

4. Let f be a periodic function of period p, i.e., for all $x \in \mathbb{R}$, $f(x+p) = f(x)$.

   (a) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x)dx = \int_a^{a+p} f(x)dx.$$

   Hint: Write $\int_a^{a+p} f(x)dx$ as the sum of two integrals (a to p and p to $a+p$) and make an appropriate change of variables.

   (b) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x+a)dx = \int_0^p f(x)dx.$$

   (c) Interpret these identities graphically.

5. pg. 76 #3.2.1, #3.2.2.