MTH 352/652: Homework #7

Due Date: March 22, 2024

Note: In this homework you are asked to plot contour plots of solutions using computer software. Your plots should printed and included in your homework preferably at the same location as the problem.

1 Problems for Everyone

1. Consider the following initial boundary value problem

$$
\begin{aligned}
u_t &= u_{xx}, \\
u(0,x) &= \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1-x & \frac{1}{2} \le x \le 1 \end{cases}, \\
u_x(t,0) &= 0, \\
u_x(t,1) &= 0.
\end{aligned}
$$

   (a) Find the solution to this PDE using Fourier series.

   (b) Calculate the equilibrium distribution, i.e., the $t \to \infty$ limit.

   (c) Using Mathematica or some other software, plot an approximation of the solution at times $t = 0, 1, 5, 10$ using the first 20 terms in the Fourier series.

   (d) Using Mathematica or some other software, plot an approximation of the contour plot of the solution. Your contour plot should be on the domain $t \in [0, 10]$, $x \in [0, 1]$ and use the first 20 terms in the Fourier series.

2. Consider the following initial boundary value problem

$$
\begin{aligned}
u_t &= u_{xx}, \\
u(0,x) &= \begin{cases} x & |x| \le 1 \\ 0 & |x| > 1 \end{cases}, \\
u(t,-2) &= 0, \\
u(t,2) &= 0.
\end{aligned}
$$

   (a) Find the solution to this PDE using Fourier series.

   (b) Calculate the equilibrium distribution, i.e., the $t \to \infty$ limit.

   (c) Using Mathematica or some other software, plot an approximation of the solution at times $t = 0, 1, 5, 10$ using the first 20 terms in the Fourier series.

   (d) Using Mathematica or some other software, plot an approximation of the contour plot of the solution. Your contour plot should be on the domain $t \in [0, 10]$, $x \in [0, 1]$ and use the first 20 terms in the Fourier series.

3. pg. 139, #4.1.9, #4.1.12-#4.1.13
4. The cable equation with Dirichlet boundary conditions is given by the following PDE

$$\begin{aligned} u_t &= \gamma u_{xx} - \alpha u, \\ u(0,x) &= f(x), \\ u(t,0) &= 0, \\ u(t,1) &= 0, \end{aligned}$$

where $\alpha, \gamma > 0$ are constants.

   (a) By making the change of variables $v = e^{-\alpha t}u$, show that if u solves the cable equation with Dirichlet boundary conditions then v solves the following heat equation

$$\begin{aligned} v_t &= \gamma v_{xx} \\ v(0,x) &= f(x) \\ v(t,0) &= 0 \\ v(t,1) &= 0 \end{aligned}$$

   (b) Using Fourier series, find the general solution to the cable equation with Dirichlet boundary conditions.
5. The convection-diffusion equation is given by $u_t + cu_x = \gamma u_{xx}$, where $c, \gamma > 0$.
   (a) Show that $v(t,x) = u(t, x + ct)$ solves the heat equation.
   (b) What is the physical interpretation of this change of variables?
6. The lossy convection-diffusion equation is given by $u_t = \gamma u_{xx} + cu_x - \alpha u$, where $\gamma, c, \alpha > 0$. By making an appropriate transformation, show that the the lossy convection-diffusion equation can be transformed to the heat equation.
7. For the following problems, write down the solutions to the following initial-boundary value problems for the wave equation in the form of a Fourier series. Then, using Mathematica or some other software, plot an approximation of the contour plot of the solution. Your contour plot should be on an appropriate spatial domain and the temporal axis should cover at least two periods of the solution.
   (a) $u_{tt} = u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 1$, $u_t(0,x) = 0$;
   (b) $u_{tt} = 2u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 0$, $u_t(0,x) = 1$;
   (c) $u_{tt} = 3u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = \sin^3(x)$, $u_t(0,x) = 0$;
   (d) $u_{tt} = 2u_{xx}$, $u_x(t,0) = u_x(t,2\pi) = 0$, $u(0,x) = -1$, $u(0,x) = 1$;
   (e) $u_{tt} = u_{xx}$, $u_x(t,0) = u_x(t,1) = 0$, $u(0,x) = x(1-x)$, $u_t(0,x) = 0$.
8. Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$ on the interval $0 < x < L$ satisfying Dirchlet boundary conditions, i.e., $u(t,0) = u(t,L) = 0$. The total energy of u at time t is

$$E(t) = \int_0^L \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx.$$

   (a) Prove that if u solves the wave equation with Dirichlet boundary conditions then $E(t)$ is a constant in time.
   (b) Prove that the only solution to the equation $v_{tt} = c^2 v_{xx}$ with Dirichlet boundary conditions and initial conditions $v(t,0) = 0$, $v_t(0,x) = 0$ is the trivial solution $v(t,x) = 0$.
   (c) Prove that there is at most one solution $u(t,x)$ to the equation $u_{tt} = c^2 u_{xx}$ with Dirchlet boundary conditions and initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$.