Lecture 9: Heat Equation

$$u_t = \gamma u_{xx}$$
$$u(0,x) = f(x)$$

Dirichlet Boundary Conditions

$$u(t,0) = u(t,\ell) = 0$$

Properties:

1. Let $E(t) = \int_0^{\ell} u(t,x)^2\,dx$

$$\Rightarrow \frac{dE}{dt} = \int_0^{\ell} 2u\cdot u_t\,dx$$
$$= \int_0^{\ell} 2\cdot u\cdot u_{xx}\,dx$$
$$= 2\cancel{u}^{0}u_x\Big|_0^{\ell} - \int_0^{\ell} 2u_x^2\,dx$$
$$\Rightarrow \frac{dE}{dt} \leq 0.$$

$\Rightarrow$ Solutions are unique.

2. Look for separable solutions

$$u(t,x) = TX$$
$$\Rightarrow T'X = \gamma T X''$$
$$\Rightarrow \frac{T'}{\gamma T} = \frac{X''}{X} = -\omega^2$$
$$\Rightarrow T = \exp(-\gamma\omega^2 t),\ X = A\cos(\omega x) + B\sin(\omega x)$$

Boundary conditions imply

$$u(t,0) = 0 \Rightarrow 0 = A$$
$$u(t,\ell) = 0 \Rightarrow 0 = B\sin(\omega\ell)$$
$$\Rightarrow \omega = \frac{n\pi}{\ell}$$

The generic solution is of the form

$$u(t,x)=\sum_{n=1}^{\infty} b_n \exp\left(\frac{-n^2\pi^2\gamma t}{\ell^2}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow u(0,x)=f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = b_n\int_0^{\ell}\sin^2\left(\frac{n\pi x}{\ell}\right)dx = b_n \ell/2$$

$$\Rightarrow b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

Neumann Boundary Conditions

$$u_t = u_{xx}$$

$$\left.\begin{aligned} u_x(t,0)&=0\\ u_x(t,1)&=0\end{aligned}\right\} \text{Insulated Boundary}$$

$$u(0,x)=-x(x-1)=-x^2+x$$

Seperation of variables yields

$$u(t,x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x)$$

$$\Rightarrow a_0 = 2\int_0^1(-x^2+x)dx = 1/3$$

$$\begin{aligned} a_n &= 2\int_0^1(-x^2+x)\cos(n\pi x)dx\\ &= \frac{-2}{n^2\pi^2}(1+(-1)^n)\\ &= \frac{2}{n^2\pi^2}((-1)^n-1)\end{aligned}$$

$$\Rightarrow u(t,x)=\frac{1}{6}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\pi^2 t}}{(2n-1)^2}\cos((2n-1)x)$$