Lecture 13: Numerical Solutions to the Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(t,0) = \alpha(t)$$

$$u(t,l) = \beta(t)$$

$$u(0,x) = f(x)$$

$$u_t(0,x) = g(x)$$

Discretization:

$$t_j = j\Delta t, \quad x_i = i\Delta x$$

$$\Rightarrow u_{tt} = \frac{u_{j+1,i} - 2u_{j,i} + u_{j-1,i}}{\Delta t^2} + O(\Delta t^2)$$

$$u_{xx} = \frac{u_{j,i+1} - 2u_{j,i} + u_{j,i-1}}{\Delta x^2} + O(\Delta x^2)$$

$$\Rightarrow \boxed{u_{j+1,i} = \frac{\Delta t^2}{c^2 \Delta x^2}\left(u_{j,i+1} - 2u_{j,i} + u_{j,i-1}\right) + 2u_{j,i} - u_{j-1,i}}$$

The CFL condition is

$$\boxed{\frac{\Delta t}{c\Delta x} < 1}$$

Initialization:

The above scheme does not work when $j=1$. One idea is to use the derivative to approximate $u_{2,i}$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{u_{2,i} - u_{1,i}}{\Delta t} + O(\Delta t) = g(x_i)$$

$$\rightarrow u_{2,i} = \Delta t\, g(x_i) + u_{1,i}$$

However this is an $\mathcal{O}(\Delta t)$ approximation, not $\mathcal{O}(\Delta t^2)$. As an alternative, we have that:

$$\frac{u(\Delta t, x_i) - u(0, x_i)}{\Delta t} = \frac{\partial u}{\partial t}(0, x_i) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(0, x_i)\Delta t + \mathcal{O}(\Delta t^2)$$

$$= g(x_i) + \frac{c^2}{2}\frac{\partial^2 u}{\partial x^2}(0, x_i)\Delta t + \mathcal{O}(\Delta t^2)$$

$$= g(x_i) + \frac{c^2}{2} f''(x_i)\Delta t + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \boxed{U_{2,i} = U_{1,i} + \Delta t\, g(x_i) + c^2\frac{\Delta t^2}{2} f''(x_i)}$$

In vector notation

$$\boxed{\begin{aligned}
\vec{U}_1 &= f(x) \\
\vec{U}_2 &= \vec{U}_1 + \Delta t\, g(x) + \frac{c^2\Delta t^2}{2} f''(x) \\
\vec{U}_{j+1} &= \frac{\Delta t^2}{c^2\Delta x^2} D2\,\vec{U}_j + 2\vec{U}_j - \vec{U}_{j-1} \\
D2 &= \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}
\end{aligned}}$$