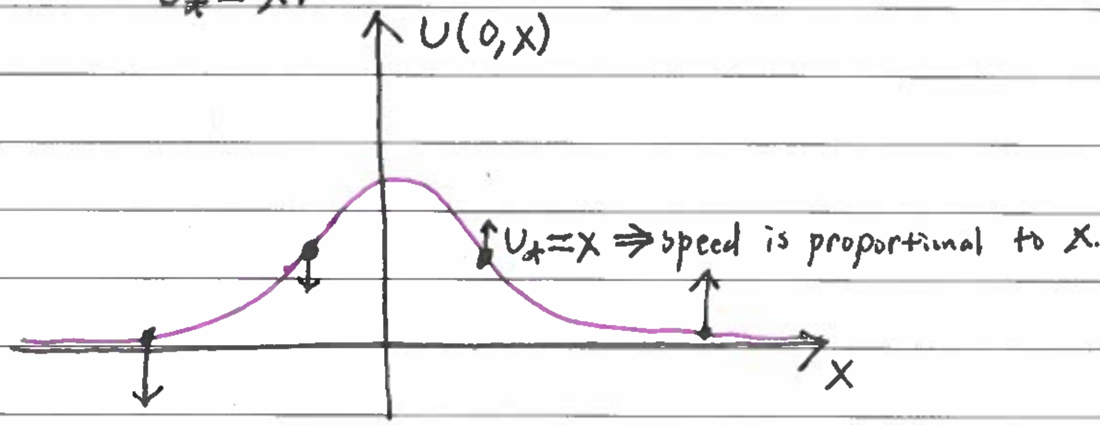
Lecture 3: Linear Transport Equation

"Boring" PDEs

1. What is the general solution of

$$U_t = x?$$



Integrate with respect to t:

$$U(t,x) = xt + f(x),$$

where $f(x)$ is an arbitrary function.

$\Rightarrow U(0,x) = U_0 = f(x)$ is the initial profile.

2. Solve the following:

$$U_t + U^2 = 0$$
$$U(0,x) = f(x)$$

We can seperate variables

$$\frac{\partial U}{\partial t} = -U^2$$

$$\Rightarrow -\int_{f(x)}^{U} \frac{1}{U^2}\, dU = \int_0^t dt$$

$$\Rightarrow \frac{1}{U} - \frac{1}{f(x)} = t$$

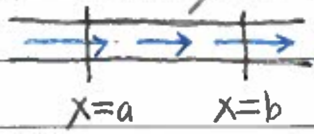
$$\Rightarrow U = \frac{f(x)}{f(x)t + 1}$$

For a fixed value of x, this function blows up at a critical time $\tau(x) = -1/f(x)$

$\Rightarrow$ If $f(x) > 0$, then $\lim_{t\to\infty} U(t,x) = 0$.

Conservation Laws

$u(t,x) \to$ density of some physical quantity



$\leftarrow$ flow of fluid, cars, people, etc.

$$\frac{d}{dt}\underbrace{\int_a^b u(t,x)\,dx}_{\text{total amount of stuff in interval } [a,b].} = \underbrace{F(a)-F(b)}_{\text{rule for flux comes from modeling/physics}} \qquad \text{(Integral Form)}$$

$$\Rightarrow \int_a^b u_t(t,x)\,dx = -\int_a^b \frac{d}{dx} F(u(t,x))\,dx$$

$$\Rightarrow \int_a^b u_t(t,x)\,dx = -\int_a^b F'(u(t,x))\,u_x(t,x)\,dx$$

$$\Rightarrow \int_a^b \left(u_t(t,x) + F'(u(t,x))\,u_x(t,x)\right)dx = 0$$

As $b-a \to 0$ we obtain the following PDE

$$u_t + F'(u(t,x))\,u_x(t,x) = 0$$

Example:

$$u_t + c u_x = 0 \qquad \text{(transport equation)}$$

$$u(0,x) = f(x)$$

Assume $f(x)$ satisfies $\lim_{|x|\to\infty} f(x) = 0$ and $\lim_{|x|\to\infty} u(t,x) = 0$. Total density is conserved:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(t,x)\,dx = \int_{-\infty}^{\infty} u_t\,dx$$

$$= \int_{-\infty}^{\infty} -c u_x\,dx$$

$$= -c\left(u(\infty) - u(-\infty)\right)$$

$$= 0$$

Change Variables!

$z = x - ct$

$\tau = t$

$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -c\frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}$$

$$\Rightarrow u_t + c u_x = u_\tau = 0$$

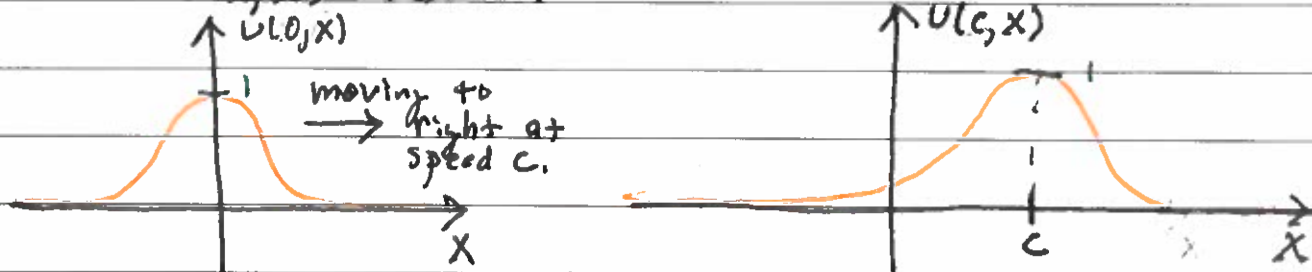
$$\Rightarrow u(\tau, z) = g(z) \Rightarrow u(t, x) = g(x - ct)$$

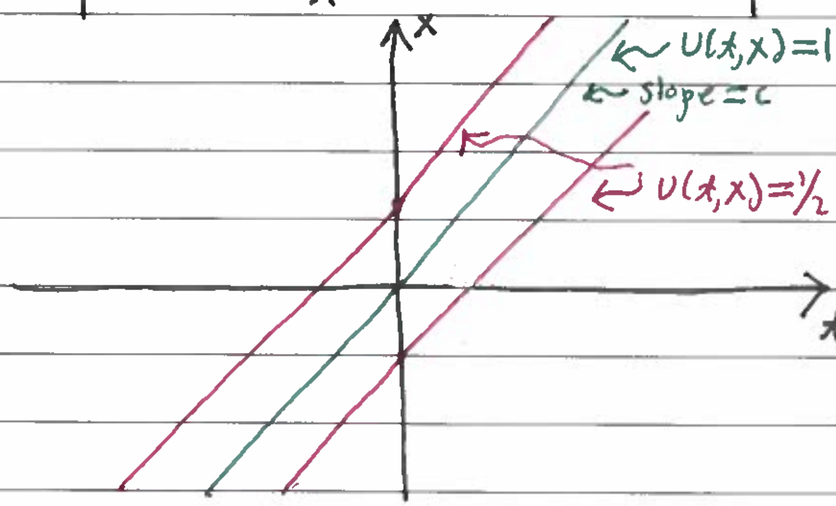
To satisfy initial conditions we have that

$$u(0, x) = g(x) = f(x)$$

Therefore,

$$u(t, x) = f(x - ct)$$





Since $f(0)=1$, along the curve $x - ct = 0 \Rightarrow x = ct$, $u(t,x) = 1$.
The lines $x = ct + b$ are called characteristic curves. On the characteristic curves, u is constant.

Damped Transport Equation

Solve,

$$u_t + cu_x + au = 0, \quad u(0,x) = f(x)$$

Let $z = x - ct$, $\tau = t$. Therefore,

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$\Rightarrow \quad u_\tau + au = 0$

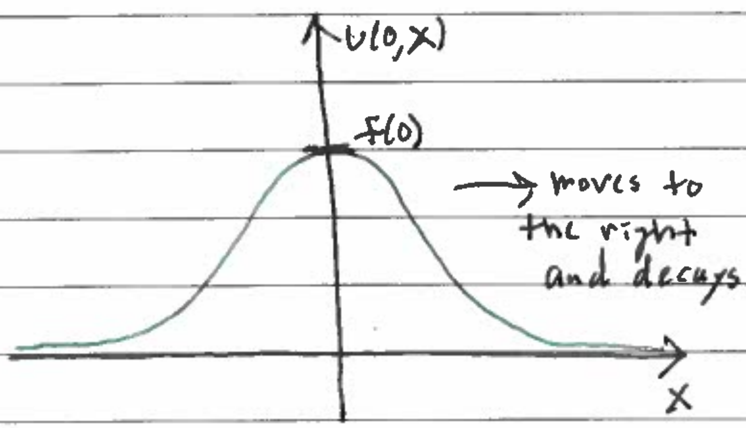
$\Rightarrow \quad u_\tau = -au$

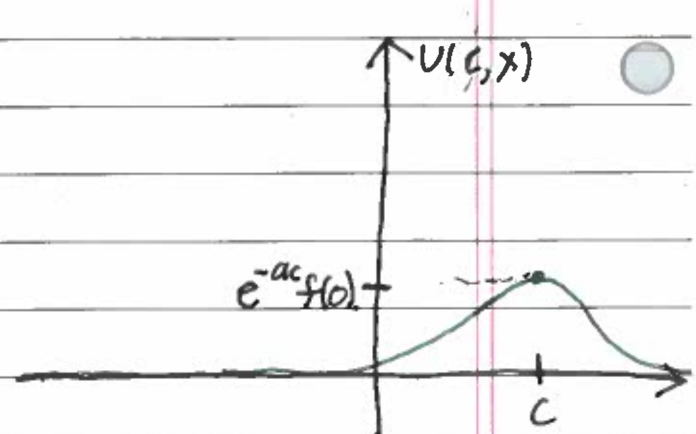
$\Rightarrow \quad u(\tau, z) = g(z)e^{-a\tau}$

$\Rightarrow \quad u(t,x) = g(x-ct)e^{-at}$

Initial conditions imply

$$u(t,x) = f(x-ct)e^{-at}$$





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