Lecture 5: Fourier Series

L^2-Inner Product

We want to write
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

* The operation $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V if

1. $\langle a\vec{v}, \vec{w} \rangle = a \langle \vec{v}, \vec{w} \rangle$
2. $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$
3. $\langle \vec{v}, \vec{w} + \vec{z} \rangle = \langle \vec{v}, \vec{w} \rangle + \langle \vec{v}, \vec{z} \rangle$

The L^2 inner product on $[-\pi, \pi]$ is defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$$
$$\Rightarrow \|f\| = \langle f, f \rangle^{1/2} = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx}.$$

Orthogonality

1. $\langle \cos(mx), \cos(nx) \rangle = 0$ if $m \neq n$
2. $\langle \sin(mx), \sin(nx) \rangle = 0$ if $m \neq n$
3. $\langle \cos(mx), \sin(mx) \rangle = 0$
4. $\langle \cos(mx), \cos(mx) \rangle = 1$
5. $\langle \sin(mx), \sin(mx) \rangle = 1$
6. $\langle 1, 1 \rangle = 2$

$\Rightarrow$ The trigonometric functions form an orthogonal system.
This is very similar to a basis.

Example:

$f(x) = x$

$$\Rightarrow x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Using Orthogonality:

1. $\langle x, 1 \rangle = \frac{a_0}{2} \langle 1, 1 \rangle = a_0$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0$$

2. $\langle x, \cos(mx) \rangle = a_m \langle \cos(mx), \cos(mx) \rangle$

$$\Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(mx) \, dx = 0.$$

3. $\langle x, \sin(mx) \rangle = b_m \langle \sin(mx), \sin(mx) \rangle$

$$\Rightarrow b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(mx) \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin(mx) \, dx$$

$$= \frac{2}{\pi} \left[\frac{-x \cos(mx)}{m} \Bigg|_0^{\pi} + \cancelto{0}{\int_0^{\pi} \frac{\cos(mx)}{m} \, dx} \right]$$

$$= \frac{-2}{\pi} \frac{\cos(m\pi)}{m}$$

$$= \frac{2(-1)^{m+1}}{m}$$

Therefore,

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots \right)$$

Convergence Issues

$$f(x) = x \sim 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right)$$

1. $f(0) = 0 = 2\left(\sin(0) - \frac{\sin(2\cdot 0)}{2} + \frac{\sin(3\cdot 0)}{3} + \dots\right)$ ✓

2. $f(\pi/2) = 2\left(\sin(\pi/2) - \frac{\sin(\pi)}{2} + \frac{\sin(3\cdot\pi/2)}{3} + \dots\right)$

$$= 2\left(1 - 1/3 + 1/5 + \dots\right)$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)}$$

$$= \pi/2$$ ✓

3. $f(\pi) = \pi = 2\left(\sin(\pi) - \frac{\sin(2\pi)}{2} + \frac{\sin(3\pi)}{3} + \dots\right)$

$$\Rightarrow \pi = 0$$:(

The issue is that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

is a 2π-periodic function while $f(x) = x$ is not.

Periodic Extensions

Lemma - If $f(x)$ is any function defined for $-\pi < x < \pi$, then there is a unique 2π-periodic function $\hat{f}$, known as the 2π periodic extension of that satisfies $\hat{f}(x) = f(x)$ for all $-\pi < x < \pi$

proof

Let $x \in \mathbb{R}$. There exists $m \in \mathbb{Z}$ such that

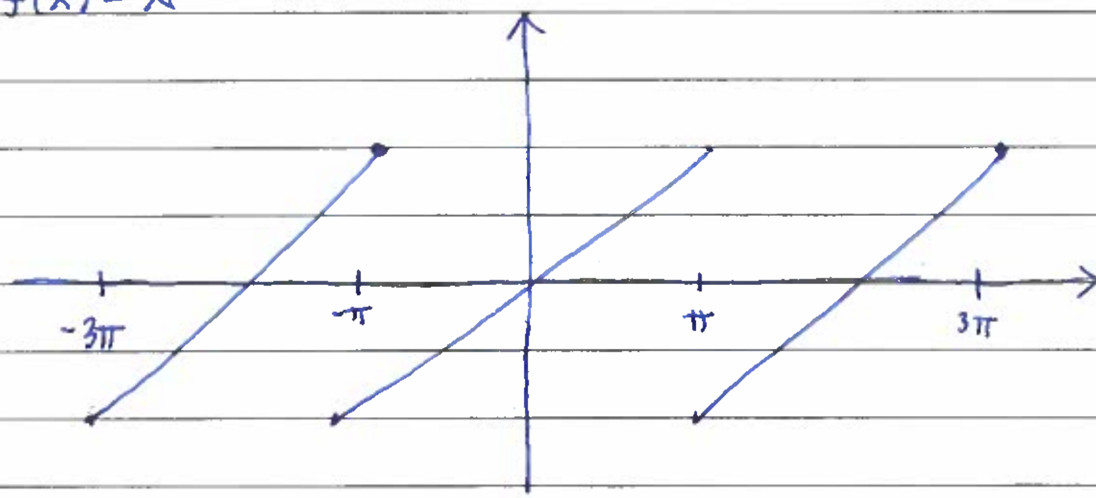
$$(2m-1)\pi \le x \le (2m+1)\pi$$

Define,

$$\tilde{f}(x) = f(x - 2m\pi)$$

Example:

$f(x) = x$



Theorem - If $\hat{f}(x)$ is a 2π-periodic, piecewise $\mathcal{C}^1$ function then at any x:

1. $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ (if $\tilde{f}$ is continuous at x)

2. $\frac{1}{2}\left[\hat{f}(x^+) + \tilde{f}(x^-)\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ (if $\hat{f}$ has a jump discontinuity at x)

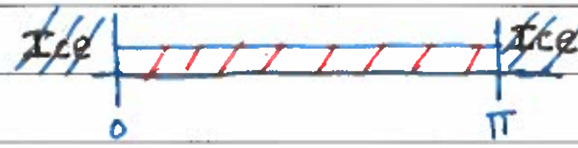
Cooling of a Rod:

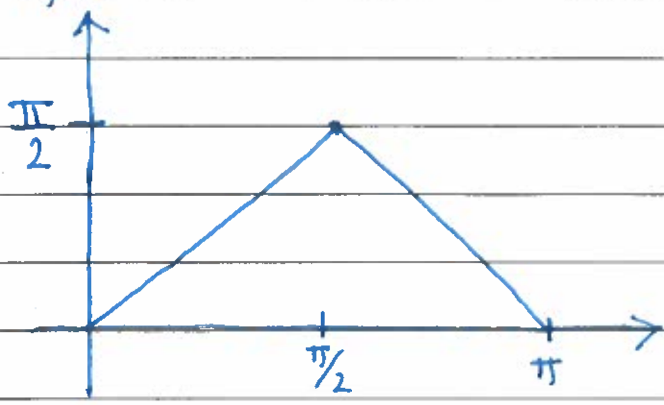
$u_t = u_{xx}$

$u(0,x) = f(x) = \pi/2 - |x - \pi/2|$

$u(t,0) = 0$

$u(t,\pi) = 0$





Guess:

$u(t,x) = T \cdot X$

$\Rightarrow T'X = TX''$

$\Rightarrow \dfrac{T'}{T} = \dfrac{X''}{X} = \lambda$

The $\lambda < 0$ are the only ones that yield nonzero eigenfunctions.

Let $\lambda = -w^2$. Therefore,

$X = A\cos(wx) + B\sin(wx)$, $T = e^{-w^2 t}$

$u(t,0) = 0 \Rightarrow A = 0.$

$u(t,\pi) = 0 \Rightarrow B\sin(w\pi) = 0$

Therefore, we need

$w = n \in \mathbb{N}.$

The generic solution is therefore

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

$$\Rightarrow u(0,x) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Consequently, by orthogonality:

$$\int_0^{\pi} f(x)\sin(nx)\,dx = b_n \int_0^{\pi} \sin^2(nx)$$
$$= b_n \cdot \pi/2$$

Therefore,

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} \sin(nx)\,x\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)(\pi - x)\,dx$$
$$= \frac{2}{\pi}\left(\frac{-\cos(nx)}{n}x \Big|_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos(nx)}{n}\,dx \right)$$
$$+ \frac{2}{\pi}\left(\frac{-\cos(nx)(\pi - x)}{n} \Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\cos(nx)}{n}\,dx \right)$$
$$= \frac{2}{\pi}\left(\frac{-\cos(n\pi/2)\pi}{2n} + \frac{\sin(n\pi/2)}{n^2} + \frac{\cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} \right)$$
$$= \frac{4}{\pi}\,\frac{\sin(n\pi/2)}{n^2}$$

$$\Rightarrow b_n = \frac{4}{\pi}\begin{cases} 1/n^2, & n = 1, 5, 9, \ldots \\ -1/n^2, & n = 3, 7, 11, \ldots \\ 0, & n = \text{even}. \end{cases}$$

$$\boxed{u(t,x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-(2n-1)^2 t} \sin((2n-1)x)}$$