**MTH 225**
**Spring 2025**
**Exam 2**
**03/31/25**

**Name (Print):** Key

This exam contains 8 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 15 | |
| 8 | 15 | |
| Total: | 100 | |

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   **C** **I** If $A \in M_{n\times n}(\mathbb{R})$ has distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ then, with respect to the standard inner product, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ .

   **C** **I** If $A \in M_{n\times n}(\mathbb{C})$ then $\ker(A) \cap \operatorname{im}(A) = \{0\}$.

   **C** **I** If $A, B \in M_{n\times n}(\mathbb{C})$ are unitary then $A + B$ is unitary.

   **C** **I** If $\mathbf{v}$ is an eigenvector of the matrices $A, B \in M_{n\times n}(\mathbb{C})$ then it is an eigenvector of their sum $A + B$.

   **C** **I** If $\lambda$ is an eigenvalue of the matrices $A, B \in M_{n\times n}(\mathbb{C})$ then it is an eigenvalue of their sum $A + B$.

2. (10 points) **Short Answer:** Let $T : V \mapsto V$ be a linear transformation on a vector space $V$. Write down the definition of what it means for $\lambda$ to be an eigenvalue of $T$. **Hint:** $T$ does not necessarily have a matrix representation.

If $\lambda$ is an eigenvalue of $T$ then there exists $\vec{v} \neq 0$ such that $T(\vec{v}) = \lambda \vec{v}$.

3. (10 points) Let $A \in M_{n\times n}(\mathbb{C})$ and suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ with corresponding eigenvector $\mathbf{v} \in \mathbb{C}^n$. If $B \in M_{n\times n}(\mathbb{C})$ is given by

$$B = a^2A^2 + 2abA + b^2I,$$

where $I$ denotes the identity matrix in $M_{n\times n}(\mathbb{C})$ and $a, b \in \mathbb{C}$, show that $\mathbf{v}$ is an eigenvector of $B$ and find its corresponding eigenvalue.

$$\begin{aligned} B\vec{v} &= a^2A^2\vec{v} + 2abA\vec{v} + b^2\vec{v} \\ &= a^2\lambda^2\vec{v} + 2ab\lambda\vec{v} + b^2\vec{v} \\ &= (a^2\lambda^2 + 2ab\lambda + b^2)\vec{v} \end{aligned}$$

There $\vec{v}$ is an eigenvector with eigenvalue $a^2\lambda^2 + 2ab\lambda + b^2$.

4. (10 points) Determine if the function $\langle \cdot, \cdot \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \mapsto \mathbb{C}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_2 + v_2 w_1$$

is an inner product on $\mathbb{C}^2$. If it is in an inner product, prove it. If it is not an inner product, provide a counterexample that violates one of the properties of an inner product.

This is not an inner product since

$\left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle = -2 < 0.$

5. (10 points) Let $\beta = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$ defined by

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

are orthonormal with respect to the standard inner product on $\mathbb{R}^3$. If $\mathbf{v} \in \mathbb{R}^3$ is given by

$$\mathbf{v} = \begin{bmatrix}1\\-1\\1\end{bmatrix},$$

find the coordinates of $\mathbf{v}$ with respect to the $\beta$ basis, i.e., find $[\mathbf{v}]_\beta$.

$$[v]_\beta = \begin{bmatrix}\langle \vec{v}, \vec{u}_1\rangle \\ \langle \vec{v}, \vec{u}_2\rangle \\ \langle \vec{v}, \vec{u}_3\rangle\end{bmatrix} = \begin{bmatrix}1/\sqrt{3} \\ 0 \\ -4/\sqrt{6}\end{bmatrix}$$

6. (15 points) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are orthonormal vectors with respect to the standard complex inner product and $\langle \mathbf{u}, \mathbf{v} \rangle = i$. Compute and simplify the following:

$$\langle \mathbf{u} + i\mathbf{v}, \mathbf{u} - i\mathbf{v} \rangle.$$

$$\langle \vec{u} + i\vec{v}, \vec{u} - i\vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, -i\vec{v} \rangle + \langle i\vec{v}, \vec{u} \rangle + \langle i\vec{v}, -i\vec{v} \rangle$$

$$= \|\vec{u}\|^2 - i\langle \vec{u}, \vec{v} \rangle - i\langle \vec{v}, \vec{u} \rangle + (-i)(-i)\langle \vec{v}, \vec{v} \rangle$$

$$= 1 - i \cdot i - i \cdot \overline{\langle \vec{u}, \vec{v} \rangle} - \|\vec{v}\|^2$$

$$= 1 + 1 + i^2 - 1$$

$$= 0.$$

7. (15 points) Let $V = P_1(\mathbb{R})$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)x\,dx.$$

With respect to this inner product, find orthonormal $p_1(x), p_2(x) \in V$ such that

$$\operatorname{span}\{p_1(x), p_2(x)\} = \operatorname{span}\{1, x\}.$$

- Let $\vec{w}_1 = 1$. Therefore,

$$\langle \vec{w}_1, \vec{w}_1 \rangle = \int_0^1 x\,dx = 1/2$$

- Let $p_1(x) = \sqrt{2}$.
- Let $\vec{w}_2 = x - \langle p_1(x), x \rangle p_1$. Consequently,

$$\vec{w}_2 = x - \left(\int_0^1 \sqrt{2}\,x^2\,dx\right)\sqrt{2}$$

$$= x - 2/3.$$

- $$\|\vec{w}_2\|^2 = \int_0^1 (x - 2/3)^2 x\,dx$$

$$= \int_0^1 \left(x^3 - \frac{4}{3}x^2 + \frac{4}{9}x\right)dx$$

$$= 1/4 - 4/9 + 2/9$$

$$= \frac{9 - 16 + 8}{36}$$

$$= 1/36$$

$$\Rightarrow \|\vec{w}_2\| = 1/6$$

Therefore,

$$p_2(x) = 6\,(x - 2/3)$$

8. (15 points) Suppose $A \in M_{n\times n}(\mathbb{C})$.

(a) (5 points) **Short Answer:** Write down the definition of $\ker(A)$, $\mathrm{im}(A)$, $\mathrm{nul}(A)$, and $\mathrm{rank}(A)$.

$\ker(A) = \{\vec{v} \in \mathbb{C}^n : A\vec{v} = 0\}$

$\mathrm{im}(A) = \{\vec{w} \in \mathbb{C}^n :$ there exists $\vec{v} \in \mathbb{C}^n$ such that $A\vec{v} = \vec{w}\}$.

$\mathrm{nul}(A) = \dim(\ker(A))$

$\mathrm{rank}(A) = \dim(\mathrm{im}(A))$

(b) (5 points) **Short Answer:** State the rank-nullity theorem for $A$.

$\mathrm{rank}(A) + \mathrm{nul}(A) = n$

(c) (5 points) **Short Answer:** If $n = 4$, find a matrix $A \in M_{4\times 4}(\mathbb{R})$ such that

$$\ker(A) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \mathrm{im}(A) = \mathrm{span}\left\{ \begin{bmatrix} \pi \\ \pi^2 \\ \pi^3 \\ \pi^4 \end{bmatrix}, \begin{bmatrix} e \\ e^2 \\ e^3 \\ e^4 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} \pi & 0 & e & 0 \\ \pi^2 & 0 & e^2 & 0 \\ \pi^3 & 0 & e^3 & 0 \\ \pi^4 & 0 & e^4 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} e & 0 & \pi & 0 \\ e^2 & 0 & \pi^2 & 0 \\ e^3 & 0 & \pi^3 & 0 \\ e^4 & 0 & \pi^4 & 0 \end{bmatrix}$$