MTH 225
Homework #10

Due Date: April 16, 2025

1. Suppose that $A = U\Sigma V^*$ is a singular value decomposition of a matrix $A \in M_{n\times n}(\mathbb{C})$. Find the singular decomposition of A^*.

2. Suppose $P \in M_{n\times n}(\mathbb{C})$ is a unitary matrix and $A \in M_{n\times n}(\mathbb{C})$. Show that PA has the same singular values as A.

3. Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 3i & -3i \\ i & -i \end{bmatrix}.$$

   (a) By inspection, find the kernel of each of these matrices.

   (b) By inspection, find the image of each of these matrices.

   (c) By *not* using the Gram matrix, find the SVD for each of these matrices.

4. Prove that if $A \in M_{n\times n}(\mathbb{C})$ is a Hermitian matrix with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ then its singular values satisfy $\sigma_1 = |\lambda_1|, \sigma_2 = |\lambda_2|, \ldots, \sigma_n = |\lambda_n|$.

5. If $A, B \in M_{n\times n}(\mathbb{C})$, prove or provide a counterexample that if A and B are similar then A and B have the same singular values.

6. Give examples of $A, B \in M_{2\times 2}(\mathbb{C})$ that have:

   (a) equal singular values but distinct eigenvalues,

   (b) equal eigenvalues but distinct singular values.

7. If $A \in M_{n\times n}(\mathbb{C}$, prove or provide a counterexample that the singular values of A^2 are the squares of σ_i^2 of the singular values of A.

8. Use the Gram matrix to find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   Hint: There is not a quick way to do this problem by inspection. However, once you find $\mathbf{v}_1$ and σ_1, you can directly compute $\mathbf{u}_1$ and then use orthogonality to obtain $\mathbf{v}_2$ and thus obtain $\mathbf{u}_2$ and σ_2.

Homework #10

#1

Suppose that $A = U\Sigma V^*$ is a singular value decomposition of a matrix $A \in M_{n\times n}(\mathbb{C})$. Find the singular value decomposition of A^*.

Solution:

Since $A = U\Sigma V^*$ it follows that $A^* = V\Sigma U^*$ which is the SVD of A^*.

#2.

Suppose $P \in M_{n\times n}(\mathbb{C})$ is unitary and $A \in M_{n\times n}(\mathbb{C})$. Show that PA has the same singular values as A.

Solution:

Let $A = U\Sigma V^*$ be the SVD of A. If we let $\tilde{U} = PU$ it follows that $\tilde{U}$ is unitary and

$$PA = \tilde{U}\Sigma V^*$$

is an SVD of PA. Consequently, PA and A have the same singular values.

#3

Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3i & -3i \\ i & -i \end{bmatrix}$$

(a) By inspection, find the kernel of these matrices.

(b) By inspection, find the image of these matrices.

(c) Find the SVD of these matrices.

Solution:

(a) $\ker(A) = \operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

$\ker(B) = \{0\}$

$\ker(C) = \operatorname{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$\ker(D) = \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}\right\}.$

(b) $\operatorname{im}(A) = \operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right\}$

$\operatorname{im}(B) = \operatorname{span}\left\{\begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$

$\operatorname{im}(C) = \operatorname{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$\operatorname{im}(D) = \operatorname{span}\left\{\frac{1}{\sqrt{10}}\begin{bmatrix}3i\\i\end{bmatrix}\right\}$

(c) Case A:

$\sigma_2 = 0,\ \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

Therefore,

$\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$$\Rightarrow A\vec{v}_1 = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = \sqrt{2}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}.$$

Therefore,

$\vec{u}_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix},\ \sigma_1 = \sqrt{2}.$

From orthonormality

$\vec{u}_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}.$

Consequently, $A = U\Sigma V^*$

$$\Rightarrow A = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Case B:

$\sigma_1 = \sigma_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$B\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$B\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Therefore, $B = U \Sigma V^*$

$$\Rightarrow B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Case C:

$\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 0$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$

$$\Rightarrow C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Case D:

$\sigma_2 = 0, \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

By orthogonality, we select

$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

and thus

$$D\vec{v}_1 = \begin{bmatrix} 3i & -3i \\ i & -i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 6i/\sqrt{2} \\ 2i/\sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 3i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = 2\sqrt{5} \begin{bmatrix} 3i/\sqrt{10} \\ i/\sqrt{10} \end{bmatrix}$$

Consequently, $\sigma_1 = 2\sqrt{5}$, $\vec{u}_1 = \begin{bmatrix} 3i/\sqrt{10} \\ i/\sqrt{10} \end{bmatrix}$ and by orthogonality

$\vec{u}_2 = \begin{bmatrix} i/\sqrt{10} \\ -3i/\sqrt{10} \end{bmatrix}$

Therefore, $D = U \Sigma V^*$

$$\Rightarrow D = \begin{bmatrix} 3i/\sqrt{10} & i/\sqrt{10} \\ i/\sqrt{10} & -3i/\sqrt{10} \end{bmatrix} \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

#4

Prove that if $A \in M_{n \times n}(\mathbb{C})$ is Hermitian with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ then its singular values satisfy $\sigma_1 = |\lambda|$, $\sigma_2 = |\lambda_2|, \ldots, \sigma_n = |\lambda_n|$.

Solution:

Since A is Hermitian, by the spectral theorem for Hermitian matrices there exists unitary V such that

$$A = V \Lambda V^*,$$

where $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. Now,

$$\begin{aligned} A^*A &= V \Lambda^* V^* V \Lambda V^* \\ &= V \Lambda^* \Lambda V \\ &= V \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix} V^* \end{aligned}$$

Thus the singular values, which are square roots of the eigenvalues of A^*A, satisfy

$$\sigma_1 = |\lambda_1|, \sigma_2 = |\lambda_2|, \ldots, \sigma_n = |\lambda_n|.$$

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#5

If $A, B \in M_{n \times n}(\mathbb{C})$, prove or provide a counterexample that if A and B are similar then A and B have the same singular values.

Solution:

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of A are given by $\lambda = 1, 2$ and the eigenvalues of B are given by $\lambda = 1, 2$. The eigenvectors of A are given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Therefore

$$A = \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}^{-1}$$

Which proves A and B are similar. Now,

$$A^* A = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

Which has eigenvalues

$$\nu = \frac{5 \pm \sqrt{25 - 4 \cdot 3}}{2}$$

$$= \frac{5 \pm \sqrt{13}}{2}.$$

Consequently, the singular values of A are given by

$$\sigma_1 = \sqrt{\frac{5 + \sqrt{13}}{2}}, \quad \sigma_2 = \sqrt{\frac{5 - \sqrt{13}}{2}}.$$

However, the singular values of B are clearly $\sigma_1 = 2$, $\sigma_2 = 1$.

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#6

Give examples of $A, B \in M_{2\times 2}(\mathbb{C})$ that have
(a) equal singular values but distinct eigenvalues
(b) equal eigenvalues but distinct singular values.

Solution:
(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(b) From #5, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\blacksquare$

#7

If $A \in M_{n\times n}(\mathbb{C})$, prove or provide a counterexample that the singular values of A^2 are the squares σ_i^2 of the singular values of A.

Solution:
Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The singular values of A are $\sigma_1 = 1, \sigma_2 = 0$.
However, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has singular values $\sigma_1 = \sigma_2 = 0$.

$\blacksquare$

#8

Use the Gram matrix to find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution:

Now, $A^*A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which has eigenvalues

$$\sigma_i^2 = \frac{3 \pm \sqrt{5}}{2}$$

For $\sigma_1^2 = \frac{1}{2}(3+\sqrt{5})$ we have:

$$A^*A - \sigma_1^2 I = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}.$$

An eigenvector is

$$\vec{x} = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$$

$$\|\vec{x}\|^2 = 1 + \frac{1}{4} + \sqrt{5} + \frac{5}{4}$$
$$= \frac{5}{2} + \sqrt{5}$$
$$= \frac{5 + 2\sqrt{5}}{2}$$

$$\Rightarrow \|\vec{x}\| = \sqrt{\frac{5+2\sqrt{5}}{2}}$$

Consequently,

$$\vec{v}_1 = \frac{\sqrt{2}}{\sqrt{5+2\sqrt{5}}}\begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$$

$$A\vec{v}_1 = \frac{\sqrt{2}}{\sqrt{5+2\sqrt{5}}}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{\sqrt{2}}{\sqrt{5+2\sqrt{5}}}\begin{bmatrix} \frac{3}{2} + \frac{\sqrt{5}}{2} \\ 2 + \sqrt{5} \end{bmatrix}$$

$$\vdots$$

and so on.