MTH 225
Homework #11

Due Date: April 23, 2025

1. Suppose $A \in M_{n\times n}(\mathbb{C})$ is positive definite matrix. Prove that $\langle\cdot,\cdot\rangle_A : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}$ given by
$$\langle \mathbf{v}, \mathbf{w}\rangle_A = \langle A\mathbf{v}, \mathbf{w}\rangle$$
defines a complex inner product.

2. Let $A \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix.
   (a) Prove that A is a positive semidefinite matrix if and only if all of the eigenvalues of A are nonnegative.
   (b) Prove that A is a positive definite matrix if and only if all of the eigenvalues of A are positive.

3. If $A \in M_{n\times n}(\mathbb{C})$ is a positive definite matrix, prove that
$$\sqrt{A^{-1}} = \left(\sqrt{A}\right)^{-1}.$$

4. Let $A \in M_{2\times 2}(\mathbb{C})$ be a Hermitian matrix.
   (a) Show that A is positive semidefinite if and only if $\operatorname{tr}(A) \geq 0$ and $\det(A) \geq 0$.
   (b) Show that A is positive definite if and only if $\operatorname{tr}(A) \geq 0$ and $\det(A) > 0$.

5. Give an example of a Hermitian $A \in M_{3\times 3}(\mathbb{C})$ for which $\operatorname{tr}(A) \geq 0$ and $\det A \geq 0$, and A is not positive semidefinite.

6. Consider the following matrices
$$A = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},\ C = \begin{bmatrix} 3 & 5 \\ 5 & 10 \end{bmatrix}.$$
Show that A, B, C are positive definite, but $\operatorname{tr}(ABC) < 0$. What can you say about $\det(ABC)$?

7. Find the exponential of the following matrices
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

8. Prove that $Ae^A = e^A A$.

9. Let $A \in M_{2\times 2}(\mathbb{C})$.

   (a) Prove that
   $$A^2 - \operatorname{tr}(A)A + \det(A)I = O.$$

   (b) Prove that if $\det(A) \neq 0$ then
   $$A^{-1} = \frac{\operatorname{tr}(A)I - A}{\det(A)}.$$

   (c) Prove that if $\operatorname{tr}(A) = 0$ and $\delta = \sqrt{\det(A)} > 0$ then
   $$e^A = \cos(\delta)I + \frac{\sin(\delta)}{\delta}A.$$

   (d) Establish a similar formula when $\det(A) < 0$.

10. Consider the following matrix
   $$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

   (a) Find the SVD of the matrix. **Hint:** This problem was assigned in the last homework. Make sure you go through the solutions and know how to find the SVD without computing A^*A. You can just follow my solutions.

   (b) Compute both the left and write polar decompositions of A:
   $$A = Q\sqrt{A^*A} \text{ and } \sqrt{AA^*}Q,$$
   where Q is a unitary matrix.

11. Find a Hermitian matrix H such that $Q = \exp(iH)$.

Homework #11

#1

Suppose $A \in M_{n\times n}(\mathbb{C})$ is a positive definite matrix. Prove that $\langle \cdot, \cdot \rangle_A : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle \vec{v}, \vec{w} \rangle_A = \langle A\vec{v}, \vec{w} \rangle$$

defines a complex inner product.

Solution:

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Therefore,

1. $$\begin{aligned} \langle \vec{v} + \lambda\vec{w}, \vec{u} \rangle_A &= \langle A(\vec{v} + \lambda\vec{w}), \vec{u} \rangle \\ &= \langle A\vec{v} + \lambda A\vec{w}, \vec{u} \rangle \\ &= \langle A\vec{v}, \vec{u} \rangle + \bar{\lambda} \langle A\vec{w}, \vec{u} \rangle \\ &= \langle \vec{v}, \vec{u} \rangle_A + \bar{\lambda} \langle \vec{w}, \vec{u} \rangle_A \end{aligned}$$

2. $$\begin{aligned} \langle \vec{v}, \vec{w} \rangle_A &= \underline{\langle A\vec{v}, \vec{w} \rangle} \\ &= \overline{\langle \vec{w}, A\vec{v} \rangle} \\ &= \overline{\langle A\vec{w}, \vec{v} \rangle} \\ &= \overline{\langle \vec{w}, \vec{v} \rangle_A} \end{aligned}$$

3. $\langle \vec{v}, \vec{v} \rangle_A = \langle A\vec{v}, \vec{v} \rangle \geq 0$ and only equals zero when $\vec{v} = 0$.

By items 1-3, $\langle \cdot, \cdot \rangle_A$ is an inner product.

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#2

Let $A \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix.

(a) Prove that A is positive semidefinite if and only if the eigenvalues of A are nonnegative.

(b) Prove that A is a positive definite matrix if and only if all of the eigenvalues of A are positive.

Solution:

(a-b) By the spectral theorem for Hermitian matrices there exists orthonormal basis of eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Consequently, for all $\vec{v} \in \mathbb{C}^n$ there exists $c_1, \dots, c_n \in \mathbb{C}^n$ such that

$$\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n.$$

Therefore,

$$\langle A\vec{v}, \vec{v}\rangle = \langle c_1\lambda_1\vec{v}_1 + \dots + c_n\lambda_n\vec{v}_n, c_1\lambda_1\vec{v}_1 + \dots + c_n\lambda_n\vec{v}_n\rangle$$
$$= |c_1|^2\lambda_1 + \dots + |c_n|^2\lambda_n.$$

Consequently, $\langle A\vec{v}, \vec{v}\rangle \geq 0$ for all $\vec{v}$ if and only if $|c_i|^2\lambda_i \geq 0$ for all i which implies $\lambda_i \geq 0$ for all i. Moreover, $\langle A\vec{v}, \vec{v}\rangle = 0$ if and only if there there exists j such that $\lambda_j = 0$. ∎

#3

If A is a positive definite matrix, prove that
$$\sqrt{A^{-1}} = (\sqrt{A})^{-1}.$$

Solution:

Since A is positive definite matrix it is Hermitian with positive eigenvalues λ_i. By the spectral theorem there exists a unitary matrix U such that
$$A = U\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}U^*.$$

Consequently,
$$\sqrt{A^{-1}} = \left(U\begin{pmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{pmatrix}U^*\right)^{1/2}$$
$$= U\begin{pmatrix} 1/\sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & 1/\sqrt{\lambda_n} \end{pmatrix}U^*$$
$$= U\begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}^{-1}U^*$$
$$= \left(U\begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}U^*\right)^{-1}$$
$$= (\sqrt{A})^{-1}$$

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#4

Let $A \in M_{2\times 2}(\mathbb{C})$ be a Hermitian matrix.

(a) Show that A is positive definite if and only if $tr(A) \geq 0$ and $det(A) \geq 0$.

(b) Show that A is a positive definite matrix if and only if $tr(A) \geq 0$ and $det(A) \geq 0$.

Solution:

(a-b) We know that $tr(A) = \lambda_1 + \lambda_2$ and $det(A) = \lambda_1 \lambda_2$. Therefore, A is positive semidefinite if only if $\lambda_1, \lambda_2 \geq 0$ which is satisfied if and only if $\lambda_1 + \lambda_2 \geq 0$ and $\lambda_1 \cdot \lambda_2 \geq 0$. Furthermore, A is positive definite in addition to being positive semidefinite if and only if $\lambda_1, \lambda_2 \neq 0 \Rightarrow det(A) \neq 0$.

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#5

Give an example of a Hermitian $A \in M_{3\times 3}(\mathbb{C})$ for which $tr(A) \geq 0$ and $det(A) \geq 0$ and A is not positive semidefinite.

Solution:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ satisfies $tr(A) = 1 \geq 0$ and $det(A) = 3 \geq 0$

but is not positive definite since its eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$, $\lambda_3 = -1$.

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#6

Consider the following matrices

$$A=\begin{bmatrix}10 & 0\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix},\quad C=\begin{bmatrix}3 & 5\\ 5 & 10\end{bmatrix}.$$

Show that A,B,C are positive definite, but $tr(ABC)<0$. What can you say about $det(ABC)$?

Solution:

- $tr(A)=11$, $det(A)=10 \Rightarrow A$ is positive definite.
- $tr(B)=3$, $det(B)=1 \Rightarrow B$ is positive definite.
- $tr(C)=13$, $det(C)=5 \Rightarrow C$ is positive definite.

$$\begin{aligned} ABC &= \begin{bmatrix}10 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\ 5 & 10\end{bmatrix}\\ &= \begin{bmatrix}10 & -10\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\ 5 & 10\end{bmatrix}\\ &= \begin{bmatrix}-20 & -50\\ 7 & 15\end{bmatrix}\end{aligned}$$

$\Rightarrow tr(ABC)=-5$

$$\begin{aligned} det(ABC) &= det(A)det(B)det(C)\\ &= 10\cdot 1\cdot 5\\ &= 50. \end{aligned}$$

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#7

Find the exponential of the following matrices

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ B=\begin{bmatrix}0&-1\\1&0\end{bmatrix},\ C=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\ D=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

<u>Solution:</u>

<u>A:</u>

$tr(A)=0,\ det(A)=-1 \rightarrow \lambda_1+\lambda_2=0,\ \lambda_1\lambda_2=-1 \Rightarrow \lambda_1=1,\ \lambda_2=-1.$

<u>$\lambda_1=1$:</u>

$$Ker(A-\lambda_1 I)=Ker\left(\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\right)=span\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right\}.$$

<u>$\lambda_2=-1$:</u>

$$Ker(A-\lambda_2 I)=Ker\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)=span\left\{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}\right\}.$$

Therefore,

$$A=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}$$

$$\Rightarrow exp(A)=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}e&0\\0&e^{-1}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}$$

$$=\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}e/\sqrt{2}&e/\sqrt{2}\\e^{-1}/\sqrt{2}&-e^{-1}/\sqrt{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}(e+e^{-1})&\frac{1}{2}(e-e^{-1})\\\frac{1}{2}(e-e^{-1})&\frac{1}{2}(e+e^{-1})\end{bmatrix}$$

$$=\begin{bmatrix}\cosh(1)&\sinh(1)\\\sinh(1)&\cosh(1)\end{bmatrix}$$

B:

$tr(B)=0,\ det(B)=1 \Rightarrow \lambda_1+\lambda_2=0,\ \lambda_1\lambda_2=1 \Rightarrow \lambda_1=i,\ \lambda_2=-i.$

$\lambda_1=i$:

$$\ker(B-\lambda_1 I)=\ker\begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix}=span\left\{\begin{bmatrix} i/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}\right\}$$

$\lambda_2=-i$:

By orthogonality, $\ker(B-\lambda_2 I)=span\left\{\begin{bmatrix} i/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}\right\}$.

There fore,

$$B=\begin{bmatrix} i/\sqrt{2} & i/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix}\begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2}\\ -i/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

$$\Rightarrow \exp(B)=\begin{bmatrix} i/\sqrt{2} & i/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix} e^{i} & 0\\ 0 & e^{-i}\end{bmatrix}\begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2}\\ -i/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

$$=\begin{bmatrix} i/\sqrt{2}\,e^{i} & i/\sqrt{2}\,e^{-i}\\ 1/\sqrt{2}\,e^{i} & -1/\sqrt{2}\,e^{-i}\end{bmatrix}\begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2}\\ -i/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

$$=\begin{bmatrix} 1/2(e^{i}+e^{-i}) & i/2(e^{i}-e^{-i})\\ -i/2(e^{i}-e^{-i}) & 1/2(e^{i}+e^{-i})\end{bmatrix}$$

$$=\begin{bmatrix} \cos(1) & -\sin(1)\\ \sin(1) & \cos(1)\end{bmatrix}$$

C:

$$C^2=\begin{bmatrix} 0&0&1\\ 0&0&0\\ 0&0&0\end{bmatrix},\quad C^3=\begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\end{bmatrix}$$

There fore,

$$\exp(A)=I+C+\frac{1}{2}C^2=\begin{bmatrix} 1&1&1/2\\ 0&1&1\\ 0&0&1\end{bmatrix}.$$

D:

$$\det(\lambda I - D) = \det\left(\begin{bmatrix} \lambda & 0 & -1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}\right) = \lambda^3 - 1 \Rightarrow \lambda = 1, e^{2\pi i/3}, e^{4\pi i/3}$$
$$= 1, -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i, -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i$$

$\lambda = 1$:

$$\ker(\lambda I - D) = \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

$\lambda = e^{2\pi i/3}$:

$$\ker(\lambda I - D) = \ker\left(\begin{bmatrix} e^{2\pi i/3} & 0 & -1 \\ -1 & e^{2\pi i/3} & 0 \\ 0 & -1 & e^{2\pi i/3} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -e^{-2\pi i/3} \\ -1 & e^{2\pi i/3} & 0 \\ 0 & -1 & e^{2\pi i/3} \end{bmatrix}\right)$$
$$= \ker\left(\begin{bmatrix} 1 & 0 & -e^{-2\pi i/3} \\ 0 & e^{2\pi i/3} & -e^{-2\pi i/3} \\ 0 & -1 & e^{2\pi i/3} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -e^{-2\pi i/3} \\ 0 & 1 & -e^{-4\pi i/3} \\ 0 & -1 & e^{2\pi i/3} \end{bmatrix}\right)$$
$$= \ker\left(\begin{bmatrix} 1 & 0 & -e^{4\pi i/3} \\ 0 & 1 & -e^{2\pi i/3} \\ 0 & -1 & e^{2\pi i/3} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -e^{4\pi i/3} \\ 0 & 1 & -e^{2\pi i/3} \\ 0 & 0 & 0 \end{bmatrix}\right)$$
$$= \operatorname{span}\left\{\begin{bmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ 1 \end{bmatrix}\right\}$$

$\lambda = e^{4\pi i/3}$:

The eigenvector will be the complex conjugate

$$\ker(\lambda I - D) = \operatorname{span}\left\{\begin{bmatrix} e^{-4\pi i/3} \\ e^{-2\pi i/3} \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i \\ -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \\ 1 \end{bmatrix}\right\}$$

Therefore,

$$D = \underbrace{\begin{bmatrix} 1 & \exp(4\pi i/3) & \exp(2\pi i/3) \\ 1 & \exp(2\pi i/3) & \exp(4\pi i/3) \\ 1 & 1 & 1 \end{bmatrix}}_{V} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i & 0 \\ 0 & 0 & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \exp(4\pi i/3) & \exp(2\pi i/3) \\ 1 & \exp(2\pi i/3) & \exp(4\pi i/3) \\ 1 & 1 & 1 \end{bmatrix}^{-1}}_{V^{-1}}$$

$$\Rightarrow \exp(D) = V \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^{-1/2} e^{\sqrt{3}/2\, i} & 0 \\ 0 & 0 & e^{-1/2} e^{-\sqrt{3}/2\, i} \end{bmatrix} V^{-1}$$

#8

Prove that $Ae^A = e^A A$.

proof:

$$Ae^A = A(I + A + \tfrac{1}{2}A^2 + \dots)$$
$$= A + A^2 + \tfrac{1}{2}A^3 + \dots)$$
$$= (I + A + \tfrac{1}{2}A^2 + \dots)A$$
$$= e^A A.$$

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#9

Let $A \in M_{2\times 2}(\mathbb{C})$.

(a) Prove that
$$A^2 - tr(A)A + det(A)I = 0$$

(b) Prove that if $det(A) \neq 0$ then
$$A^{-1} = \frac{tr(A)I - A}{det(A)}$$

(c) Prove that if $tr(A) = 0$ and $\delta = \sqrt{det(A)} > 0$ then
$$e^A = \cos(\delta)I + \sin(\delta)/\delta\, A.$$

(d) Establish a similar formula when $det(A) < 0$.

Solution:

(a) If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^2 - tr(A)A + det(A)I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+bd \\ ac+dc & ad+d^2 \end{bmatrix}$$
$$+ \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$
$$= 0.$$

(b) $\left(\dfrac{tr(A)I - A}{det(A)}\right) A = \dfrac{tr(A)A - A^2}{det(A)} = \dfrac{tr(A) - tr(A)A + det(A)I}{det(A)} = I.$

(c) If $tr(A) = 0$ then

$$A^2 = -det(A)I$$
$$\Rightarrow A^2 = -\delta^2 I.$$

Consequently,

$$\begin{aligned} e^A &= I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \tfrac{1}{4!}A^4 + \dots \\ &= I + A - \tfrac{\delta^2}{2}I - \tfrac{\delta^2}{3!}A + \tfrac{\delta^4}{4!}I + \tfrac{\delta^4}{5!}A + \dots \\ &= I(1 - \delta^2/2 + \delta^4/4! + \dots) + A(1 - \delta^2/3! + \delta^4/5! + \dots) \\ &= I\cos(\delta) + A\,{}^1\!/\!_\delta(\delta - \delta^3/3! + \delta^5/5! + \dots) \\ &= \cos(\delta)I + \frac{\sin(\delta)}{\delta}A \end{aligned}$$

(d) If $det(A) < 0$, we will obtain

$$e^A = \cosh(\delta)I + \frac{\sinh(\delta)}{\delta}A.$$

#10

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the SVD of the matrix.

(b) Compute the left and right polar decomposition of A:

$$A = Q\sqrt{A^*A} \quad \text{and} \quad A = \sqrt{AA^*}\,Q$$

(c) Find a Hermitian matrix H such that

$$Q = \exp(iH).$$

Solution:

(a) Since $\ker(A^*) = \operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ it follows that $\sigma_2 = 0$ and $\vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Therefore, $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and

$$A \cdot \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = \sqrt{2}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}.$$

Consequently, $\sigma_1 = \sqrt{2}$ and $\vec{u}_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ which implies $\vec{u}_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$.

Therefore,

$$A = U\Sigma V^*$$
$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

(b) Now, $A = UV^*V\Sigma V^*$ and $A = U\Sigma U^* UV^*$ and thus

$$Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

(c) $\operatorname{tr}(Q) = 0$, $\det(A) = -1$ which implies $\lambda_{1,2} = \pm 1$.

$\underline{\lambda_1 = 1}$

$$\ker(\lambda I - A) = \ker\left(\begin{bmatrix}1-1/\sqrt{2} & -1/\sqrt{2}\\-1/\sqrt{2} & 1+1/\sqrt{2}\end{bmatrix}\right) = \ker\left(\begin{bmatrix}\sqrt{2}-1 & -1\\-1 & \sqrt{2}+1\end{bmatrix}\right)$$
$$= \operatorname{span}\left\{\begin{bmatrix}1\\\sqrt{2}-1\end{bmatrix}\right\} = \operatorname{span}\left\{\frac{1}{\sqrt{2(2-\sqrt{2})}}\begin{bmatrix}1\\\sqrt{2}-1\end{bmatrix}\right\}$$

$\underline{\lambda_2 = -1}$

$$\ker(\lambda I - A) = \ker\left(\begin{bmatrix}-1-1/\sqrt{2} & -1/\sqrt{2}\\-1/\sqrt{2} & -1+1/\sqrt{2}\end{bmatrix}\right) = \ker\left(\begin{bmatrix}\sqrt{2}+1 & 1\\1 & \sqrt{2}-1\end{bmatrix}\right)$$
$$= \operatorname{span}\left\{\begin{bmatrix}1\\-\sqrt{2}-1\end{bmatrix}\right\} = \operatorname{span}\left\{\frac{1}{\sqrt{2(2+\sqrt{2})}}\begin{bmatrix}-1\\\sqrt{2}+1\end{bmatrix}\right\}$$

$$\Rightarrow Q = \begin{bmatrix}1/\sqrt{2(2-\sqrt{2})} & -1/\sqrt{2(2+\sqrt{2})}\\(\sqrt{2}-1)/\sqrt{2(2-\sqrt{2})} & (\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})}\end{bmatrix}\begin{bmatrix}e^0 & 0\\0 & e^{i\pi}\end{bmatrix}\begin{bmatrix}1/\sqrt{2(2-\sqrt{2})} & (\sqrt{2}-1)/\sqrt{2(2-\sqrt{2})}\\-1/\sqrt{2(2+\sqrt{2})} & (\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})}\end{bmatrix}$$

Therefore,

$$H = \begin{bmatrix} 1/\sqrt{2(2-\sqrt{2})} & -1/\sqrt{2(2+\sqrt{2})} \\ (\sqrt{2}-1)/\sqrt{2(2-\sqrt{2})} & (\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2(2-\sqrt{2})} & (\sqrt{2}-1)/\sqrt{2(2-\sqrt{2})} \\ -1/\sqrt{2(2+\sqrt{2})} & (\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2(2-\sqrt{2})} & -1/\sqrt{2(2+\sqrt{2})} \\ (\sqrt{2}-1)/\sqrt{2(2-\sqrt{2})} & (\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2(2+\sqrt{2})} & -(\sqrt{2}+1)/\sqrt{2(2+\sqrt{2})} \end{bmatrix}$$

$$= \begin{bmatrix} 1/(2\sqrt{2}) & (1+\sqrt{2})/2\sqrt{6+2\sqrt{2}} \\ (1+\sqrt{2})/2\sqrt{6+2\sqrt{2}} & (3+\sqrt{2})/(2\sqrt{6+2\sqrt{2}}) \end{bmatrix}.$$

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