MTH 225
Homework #9

Due Date: April 09, 2025

1. If $U \in M_{n\times n}(\mathbb{C})$ is unitary, show that the matrices U^*, U^T and $\overline{U}$ are unitary.
2. Suppose $A \in M_{2\times 2}(\mathbb{C})$ with entries $A_{ij} \in \mathbb{C}$ is unitary. If $u_{21} = 0$, what can you say about the remaining three entries of A.
3. If $A, B \in M_{n\times n}(\mathbb{C})$, the Frobenius inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle A, B\rangle_F = \operatorname{tr}(A^* B).$$

The Pauli-spin matrices $\sigma_x, \sigma_y, \sigma_z \in M_{2\times 2}(\mathbb{C})$ are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (a) Show that $\sigma_x, \sigma_y, \sigma_z$ are all unitary matrices with respect to the standard complex inner product on $\mathbb{C}^2$.

   (b) Compute the norm of $I, \sigma_x, \sigma_y, \sigma_z$ with respect to the Frobenius inner product.

   (c) Show that $I, \sigma_x, \sigma_y, \sigma_z$ are all orthogonal with respect to the Frobenius inner product and thus form a basis for $M_{2\times 2}(\mathbb{C})$.

4. A matrix $A \in M_{n\times n}(\mathbb{C})$ is called *skew-Hermitian* if $A^* = -A$.

   (a) Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.

   (b) Prove for all $A \in M_{n\times n}(\mathbb{C})$ that there exists a Hermitian matrix $B \in M_{n\times n}(\mathbb{C})$ and a skew-Hermitian matrix $C \in M_{n\times n}(\mathbb{C})$ such that

$$A = B + C.$$

   (c) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is skew-Hermitian then iA is Hermitian.

   (d) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is Hermitian and $k \in \mathbb{N}$ then A^k is Hermitian.

   (e) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is skew-Hermitian and $k \in \mathbb{N}$ is even then A^k is Hermitian.

   (f) Prove that the eigenvalues of a skew-Hermitian matrix are pure imaginary.

   (g) Prove that the eigenvectors of a skew-Hermitian matrix are orthogonal.

   (h) Explain, given the properties proved above, why Hermitian matrices are analogous to real numbers, skew-Hermitian matrices are analogous to imaginary numbers, and generic matrices can be expressed in the form $A = \operatorname{Re}(A) + i\operatorname{Im}(A)$ for some Hermitian matrices $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$.

5. Let $A \in M_{n\times n}(\mathbb{C})$ and $B = A^*A$.

   (a) Prove that B is a Hermitian matrix.

   (b) Prove that B is *positive semi-definite* meaning $\mathbf{v}^*B\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{C}^n$.

   (c) Prove that if A is full rank then B is *positive definite* meaning $\mathbf{v}^*B\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{C}^n$ and $\mathbf{v}^*B\mathbf{v} = 0$ if and only if $\mathbf{v} = 0$.

6. Let $A \in M_{n\times n}(\mathbb{C})$ with representation $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$ as described above. Prove that A is normal if and only if
$$\mathrm{Re}(A)\mathrm{Im}(A) = \mathrm{Im}(A)\mathrm{Re}(A).$$

7. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) If A is Hermitian, show that $\langle A\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in \mathbb{C}^n$.

   (b) If $\langle A\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in \mathbb{C}^n$, prove that A is Hermitian. **Hint:** Consider the representation $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$.

8. Let $A \in M_{2\times 2}(\mathbb{C})$ be defined by
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$
where $a, b \in \mathbb{C}$.

   (a) Show that the eigenvalues of A are $a \pm ib$.

   (b) If $a, b \in \mathbb{R}$, show that A is a scalar multiple of a unitary matrix.

   (c) Show that
$$A^*A = \begin{bmatrix} |a|^2 + |b|^2 & 2i\mathrm{Im}(ba^*) \\ -2i\mathrm{Im}(ba^*) & |a|^2 + |b|^2 \end{bmatrix}.$$

   (d) Show that A is a normal matrix.

   (e) Show that the singular values of A are given by
$$\sigma_1 = (|a|^2 + |b|^2 + 2|\mathrm{Im}(a^*b)|)^{\frac{1}{2}} \text{ and } \sigma_2 = (|a|^2 + |b|^2 - 2|\mathrm{Im}(a^*b)|)^{\frac{1}{2}}.$$

Homework #9

#1

If $U \in M_{n\times n}(\mathbb{C})$ is unitary, show that U^*, U^T, and $\overline{U}$ are unitary.

Solution:

1. Since $(U^*)^* = U$ and $U^* = U^{-1}$ we have that
$$(U^*)^* \cdot U^* = U \cdot U^{-1} = I$$
and thus $(U^*)^* = (U^*)^{-1}$ proving that U^* is unitary.
2. Since $(U^T)^* = (U^*)^T = (U^{-1})^T = (U^T)^{-1}$ it follows that U^T is unitary.
3. Since $\overline{U}^* = \overline{(U^*)} = \overline{U^{-1}} = \overline{U}^{-1}$ it follows that $\overline{U}$ is unitary.

■

#2

Suppose $A \in M_{2\times 2}(\mathbb{C})$ with entries $A_{ij} \in \mathbb{C}$ is unitary. If $A_{21} = 0$ what can you say about the remaining three entries of A.

Solution:

If A is unitary and $A_{21} = 0$ then
$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Orthonormality of the columns implies $A_{12} = 0$, $|A_{11}| = 1$, and $|A_{22}| = 1$.

■

#3

If $A, B \in M_{n\times n}(\mathbb{C})$, the Frobenius inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle A, B \rangle_F = tr(A^* B)$$

The Pauli-spin matrices $\sigma_x, \sigma_y, \sigma_z \in M_{2\times 2}(\mathbb{C})$ are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) Show that $\sigma_x, \sigma_y, \sigma_z$ are unitary.

(b) Compute the norm of $I, \sigma_x, \sigma_y, \sigma_z$ with respect to the Frobenius inner product.

(c) Show that $I, \sigma_x, \sigma_y, \sigma_z$ ar all orthogonal with respect to the Frobenius inner product and thus form an orthonormal basis for $M_{2\times 2}(\mathbb{C})$.

<u>Solution:</u>

(a) Clearly the columns of $\sigma_x, \sigma_y, \sigma_z$ are orthonormal.

(b) $\|I\|^2 = \langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rangle_F = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \Rightarrow \|I\| = \sqrt{2}$.

$\|\sigma_x\|^2 = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle_F = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \Rightarrow \|\sigma_x\| = \sqrt{2}$.

$\|\sigma_y\|^2 = \langle \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rangle_F = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \Rightarrow \|\sigma_y\| = \sqrt{2}$

$\|\sigma_z\|^2 = \langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rangle_F = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \Rightarrow \|\sigma_z\| = \sqrt{2}$

(c) $\langle I, \sigma_x \rangle_F = tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0$

$\langle I, \sigma_y \rangle_F = tr\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) = 0$

$\langle I, \sigma_z \rangle_F = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 0$

$\langle \sigma_x, \sigma_y \rangle_F = tr\left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right) = 0$

$\langle \sigma_x, \sigma_z \rangle_F = tr\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 0$

$\langle \sigma_y, \sigma_z \rangle_F = tr\left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right) = 0$

■

#4

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called skew-Hermitian if $A^* = -A$.

Solution:

(a) $(A+A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$(A-A^*)^* = A^* - (A^*)^* = A^* - A = -(A-A^*)$

Therefore $A+A^*$ is Hermitian and $A-A^*$ is skew-Hermitian.

(b) $A = \frac{1}{2}(A+A^*) + \frac{1}{2}(A-A^*)$.

(c) $(iA)^* = -iA^* = iA \Rightarrow iA$ is Hermitian if A is skew-Hermitian

(d) $(A^k)^* = (A^*)^k = A^k \Rightarrow A^k$ is Hermitian if A is Hermitian.

(e) $(A^k)^* = (A^*)^k = (-A)^k = (-1)^k A^k = A^k \Rightarrow A^k$ is Hermitian if A is skew symmetric and k is even.

(f). Let λ be an eigenvalue of a skew-symmetric matrix A with corresponding eigenvector $\vec{v} \in \mathbb{C}^n$. Therefore,

$$\begin{aligned}
&\langle A\vec{v}, \vec{v}\rangle = \langle \vec{v}, -A\vec{v}\rangle \\
\Rightarrow &\langle \lambda\vec{v}, \vec{v}\rangle = \langle \vec{v}, -\lambda\vec{v}\rangle \\
\Rightarrow &\bar{\lambda}\langle \vec{v}, \vec{v}\rangle = -\lambda\langle \vec{v}, \vec{v}\rangle \\
\Rightarrow &(\bar{\lambda}+\lambda)\langle \vec{v}, \vec{v}\rangle = 0 \\
\Rightarrow &\bar{\lambda}+\lambda = 0 \\
\Rightarrow &\bar{\lambda} = -\lambda
\end{aligned}$$

Therefore, λ is pure imaginary.

(g) Let λ_1, λ_2 be distinct eigenvalues of a skew-Hermitian matrix with eigenvectors $\vec{v}_1, \vec{v}_2 \in \mathbb{C}^n$. Therefore,

$$\begin{aligned}
&\langle A\vec{v}_1, \vec{v}_2\rangle = \langle \vec{v}_1, -A\vec{v}_2\rangle \\
\Rightarrow &\langle \lambda_1\vec{v}_1, \vec{v}_2\rangle = \langle \vec{v}_1, -\lambda_2\vec{v}_2\rangle \\
\Rightarrow &-\lambda_1\langle \vec{v}_1, \vec{v}_2\rangle = -\lambda_2\langle \vec{v}_1, \vec{v}_2\rangle \\
\Rightarrow &(\lambda_1-\lambda_2)\langle \vec{v}_1, \vec{v}_2\rangle = 0 \\
\Rightarrow &\langle \vec{v}_1, \vec{v}_2\rangle = 0
\end{aligned}$$

∎

#5

Let $A \in M_{n\times n}(\mathbb{C})$ and $B = A^*A$

(a) Prove that B is Hermitian

(b) Prove that B is positive semidefinite.

(c) Prove that if A is full rank then B is positive definite.

Solution:

(a) $B^* = (A^*A)^* = A^*(A^*)^* = A^*A = B \Rightarrow B$ is Hermitian.

(b) $\vec{v}^* B\vec{v} = \vec{v}^* A^* A\vec{v} = \langle A\vec{v}, A\vec{v}\rangle \geq 0$.

(c) $\vec{v}^* B\vec{v} = \langle A\vec{v}, A\vec{v}\rangle = \|A\vec{v}\|^2 = 0 \Leftrightarrow A\vec{v} = 0 \Leftrightarrow \vec{v} \in \ker(A)$. Therefore, B is positive definite if and only if $\ker(A) = \{\vec{0}\} \Leftrightarrow A$ is full rank. ■

#7

Let $A \in M_{n\times n}(\mathbb{C})$.

(a) If A is Hermitian, show that $\langle A\vec{v}, \vec{v}\rangle$ is real for all $\vec{v} \in \mathbb{C}^n$.

(b) If $\langle A\vec{v}, \vec{v}\rangle$ is real for all $\vec{v} \in \mathbb{C}^n$, prove that A is Hermitian.

Solution:

(a) If A is Hermitian then

$$\overline{\langle A\vec{v}, \vec{v}\rangle} = \langle \vec{v}, A\vec{v}\rangle = \langle A\vec{v}, \vec{v}\rangle$$

and thus $\langle A\vec{v}, \vec{v}\rangle$ is real.

(b) Now, suppose $\langle A\vec{v}, \vec{v}\rangle$ is real for all $\vec{v} \in \mathbb{C}^n$. Therefore,

$$\overline{\langle A\vec{v}, \vec{v}\rangle} = \langle A\vec{v}, \vec{v}\rangle$$

$$\Rightarrow \langle \vec{v}, A\vec{v}\rangle = \langle A\vec{v}, \vec{v}\rangle$$

$$\Rightarrow \langle \vec{v}, (\mathrm{Re}(A) + i\,\mathrm{Im}(A))\vec{v}\rangle = \langle (\mathrm{Re}(A) + i\,\mathrm{Im}(A))\vec{v}, \vec{v}\rangle$$

$$\Rightarrow \langle \vec{v}, \mathrm{Re}(A)\vec{v}\rangle + i\langle \vec{v}, \mathrm{Im}(A)\vec{v}\rangle = \langle \mathrm{Re}(A)\vec{v}, \vec{v}\rangle - i\langle \mathrm{Im}(A)\vec{v}, \vec{v}\rangle$$

$$\Rightarrow \langle \vec{v}, \mathrm{Re}(A)\vec{v}\rangle + i\langle \vec{v}, \mathrm{Im}(A)\vec{v}\rangle = \langle \vec{v}, \mathrm{Re}(A)\vec{v}\rangle - i\langle \vec{v}, \mathrm{Im}(A)\vec{v}\rangle$$

$$\Rightarrow i\langle \vec{v}, \mathrm{Im}(A)\vec{v}\rangle = 0$$

Since this is true for all $\vec{v}$, we have $\mathrm{Im}(A) = 0 \Rightarrow A$ is Hermitian. ■

#6

Prove that $A \in M_{n\times n}(\mathbb{C})$ is normal if and only if
$$Re(A)Im(A) = Im(A)Re(A).$$

proof:

Computing, we have that
$$\begin{aligned}A^*A &= (Re(A) + i\,Im(A))^*(Re(A) + i\,Im(A))\\ &= (Re(A) - i\,Im(A))(Re(A) + i\,Im(A))\\ &= Re(A)^2 - i\,Im(A)Re(A) + i\,Re(A)Im(A) + Im(A)^2\end{aligned}$$
$$\begin{aligned}AA^* &= (Re(A) + i\,Im(A))(Re(A) + i\,Im(A))^*\\ &= (Re(A) + i\,Im(A))(Re(A) - i\,Im(A))\\ &= Re(A)^2 + i\,Im(A)Re(A) - i\,Re(A)Im(A) + Im(A)^2\end{aligned}$$

Consequently, A^*A is normal if and only if
$$\begin{aligned}&-i\,Im(A)Re(A) + i\,Re(A)Im(A) = i\,Im(A)Re(A) - i\,Re(A)Im(A)\\ &\Leftrightarrow 2i\,Re(A)Im(A) = 2i\,Im(A)Re(A)\\ &\Leftrightarrow Re(A)Im(A) = Im(A)Re(A).\end{aligned}$$

$\blacksquare$

#8

Let $A \in M_{2\times 2}(\mathbb{C})$ be defined by
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
where $a, b \in \mathbb{C}$.

(a) Show that the eigenvalues of A are $a \pm ib$.

(b) If $a, b \in \mathbb{R}$, show that A is a scalar multiple of a unitary matrix.

(c) Show that
$$A^*A = \begin{bmatrix} |a|^2 + |b|^2 & 2i\,Im(ba^*) \\ -2i\,Im(ba^*) & |a|^2 + |b|^2 \end{bmatrix}$$

(d) Show that A is a normal matrix.

(e) Show that the singular values of A are given by
$\sigma_{1,2} = (|a|^2+|b|^2 \pm 2|\mathrm{Im}(a^*b)|)^{1/2}$.

Solution:

(a) $$\lambda_{1,2} = \frac{\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}}{2}$$

$$= \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$$

$$= a \pm ib.$$

(b) This is true since the columns are orthogonal with norm $\sqrt{a^2+b^2}$.
Consequently

$$\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is unitary if $a, b \in \mathbb{R}$.

(c). $$A^*A = \begin{bmatrix} \bar{a} & \bar{b} \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} \bar{a}a+\bar{b}b & -\bar{a}b+\bar{b}a \\ -\bar{b}a+\bar{a}b & \bar{b}b+\bar{a}a \end{bmatrix}$$

$$= \begin{bmatrix} |a|^2+|b|^2 & \overline{b\bar{a}} - b\bar{a} \\ -(\overline{b\bar{a}}-b\bar{a}) & |a|^2+|b|^2 \end{bmatrix}$$

$$= \begin{bmatrix} |a|^2+|b|^2 & +2i\,\mathrm{Im}(ba^*) \\ -2i\,\mathrm{Im}(ba^*) & |a|^2+|b|^2 \end{bmatrix}$$

(d) $$AA^* = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} \bar{a} & \bar{b} \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$= \begin{bmatrix} a\bar{a}+b\bar{b} & a\bar{b}-b\bar{a} \\ b\bar{a}-a\bar{b} & b\bar{b}+a\bar{a} \end{bmatrix}$$

$$= A^*A.$$

(e.) The eigenvalues of A^*A are given by

$$\lambda_{1,2} = \frac{\mathrm{Tr}(A^*A) \pm \sqrt{\mathrm{Tr}(A^*A) - 4\det(A^*A)}}{2}$$

$$= \frac{2(|a|^2+|b|^2) \pm \sqrt{4(|a|^2+|b|^2)^2 - 4((|a|^2+|b|^2)^2 - \mathrm{Im}(ba^*)^2)}}{2}$$

$$= |a|^2+|b|^2 \pm |\mathrm{Im}(ba^*)|$$

$$\Rightarrow \sigma_{1,2} = (|a|^2+|b|^2 \pm |\mathrm{Im}(ba^*)|)^{1/2}.$$

$\blacksquare$