## MTH 225 Homework #10

## Due Date: April 16, 2025

- 1. Suppose that  $A = U\Sigma V^*$  is a singular value decomposition of a matrix  $A \in M_{n \times n}(\mathbb{C})$ . Find the singular decomposition of  $A^*$ .
- 2. Suppose  $P \in M_{n \times n}(\mathbb{C})$  is a unitary matrix and  $A \in M_{n \times n}(\mathbb{C})$ . Show that PA has the same singular values as A.
- 3. Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 3i & -3i \\ i & -i \end{bmatrix}.$$

- (a) By inspection, find the kernel of each of these matrices.
- (b) By inspection, find the image of each of these matrices.
- (c) By not using the Gram matrix, find the SVD for each of these matrices.
- 4. Prove that if  $A \in M_{n \times n}(\mathbb{C})$  is a Hermitian matrix with eigenvalues  $|\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_n|$ then its singular values satisfy  $\sigma_1 = |\lambda_1|, \sigma_2 = |\lambda_2|, \ldots, \sigma_n = |\lambda_n|$ .
- 5. If  $A, B \in M_{n \times n}(\mathbb{C})$ , prove or provide a counterexample that if A and B are similar then A and B have the same singular values.
- 6. Give examples of  $A, B \in M_{2 \times 2}(\mathbb{C})$  that have:
  - (a) equal singular values but distinct eigenvalues,
  - (b) equal eigenvalues but distinct singular values.
- 7. If  $A \in M_{n \times n}(\mathbb{C})$ , prove or provide a counterexample that the singular values of  $A^2$  are the squares of  $\sigma_i^2$  of the singular values of A.
- 8. Use the Gram matrix to find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

**Hint:** There is not a quick way to do this problem by inspection. However, once you find  $\mathbf{v}_1$  and  $\sigma_1$ , you can directly compute  $\mathbf{u}_1$  and then use orthogonality to obtain  $\mathbf{v}_2$  and thus obtain  $\mathbf{u}_2$  and  $\sigma_2$ .