MTH 225
Homework #11

Due Date: April 23, 2025

1. Suppose $A \in M_{n\times n}(\mathbb{C})$ is positive definite matrix. Prove that $\langle \cdot, \cdot \rangle_A : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}$ given by

$$\langle \mathbf{v}, \mathbf{w} \rangle_A = \langle A\mathbf{v}, \mathbf{w} \rangle$$

   defines a complex inner product.

2. Let $A \in M_{n\times n}(\mathbb{C})$ be a Hermitian matrix.
   (a) Prove that A is a positive semidefinite matrix if and only if all of the eigenvalues of A are nonnegative.
   (b) Prove that A is a positive definite matrix if and only if all of the eigenvalues of A are positive.

3. If $A \in M_{n\times n}(\mathbb{C})$ is a positive definite matrix, prove that

$$\sqrt{A^{-1}} = \left(\sqrt{A}\right)^{-1}.$$

4. Let $A \in M_{2\times 2}(\mathbb{C})$ be a Hermitian matrix.
   (a) Show that A is positive semidefinite if and only if $\operatorname{tr}(A) \geq 0$ and $\det(A) \geq 0$.
   (b) Show that A is positive definite if and only if $\operatorname{tr}(A) \geq 0$ and $\det(A) > 0$.

5. Give an example of a Hermitian $A \in M_{3\times 3}(\mathbb{C})$ for which $\operatorname{tr}(A) \geq 0$ and $\det A \geq 0$, and A is not positive semidefinite.

6. Consider the following matrices

$$A = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 3 & 5 \\ 5 & 10 \end{bmatrix}.$$

   Show that A, B, C are positive definite, but $\operatorname{tr}(ABC) < 0$. What can you say about $\det(ABC)$?

7. Find the exponential of the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

8. Prove that $Ae^A = e^A A$.

9. Let $A \in M_{2\times 2}(\mathbb{C})$.

   (a) Prove that
   $$A^2 - \operatorname{tr}(A)A + \det(A)I = O.$$

   (b) Prove that if $\det(A) \neq 0$ then
   $$A^{-1} = \frac{\operatorname{tr}(A)I - A}{\det(A)}.$$

   (c) Prove that if $\operatorname{tr}(A) = 0$ and $\delta = \sqrt{\det(A)} > 0$ then
   $$e^A = \cos(\delta)I + \frac{\sin(\delta)}{\delta}A.$$

   (d) Establish a similar formula when $\det(A) < 0$.

10. Consider the following matrix
   $$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

   (a) Find the SVD of the matrix. **Hint:** This problem was assigned in the last homework. Make sure you go through the solutions and know how to find the SVD without computing A^*A. You can just follow my solutions.

   (b) Compute both the left and write polar decompositions of A:
   $$A = Q\sqrt{A^*A} \text{ and } \sqrt{AA^*}Q,$$
   where Q is a unitary matrix.

11. Find a Hermitian matrix H such that $Q = \exp(iH)$.