MTH 225
Homework #9

Due Date: April 09, 2025

1. If $U \in M_{n\times n}(\mathbb{C})$ is unitary, show that the matrices U^*, U^T and $\overline{U}$ are unitary.
2. Suppose $A \in M_{2\times 2}(\mathbb{C})$ with entries $A_{ij} \in \mathbb{C}$ is unitary. If $u_{21} = 0$, what can you say about the remaining three entries of A.
3. If $A, B \in M_{n\times n}(\mathbb{C})$, the Frobenius inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle A, B \rangle_F = \operatorname{tr}(A^* B).$$

The Pauli-spin matrices $\sigma_x, \sigma_y, \sigma_z \in M_{2\times 2}(\mathbb{C})$ are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (a) Show that $\sigma_x, \sigma_y, \sigma_z$ are all unitary matrices with respect to the standard complex inner product on $\mathbb{C}^2$.
   (b) Compute the norm of $I, \sigma_x, \sigma_y, \sigma_z$ with respect to the Frobenius inner product.
   (c) Show that $I, \sigma_x, \sigma_y, \sigma_z$ are all orthogonal with respect to the Frobenius inner product and thus form a basis for $M_{2\times 2}(\mathbb{C})$.

4. A matrix $A \in M_{n\times n}(\mathbb{C})$ is called *skew-Hermitian* if $A^* = -A$.
   (a) Prove for all $A \in M_{n\times n}(\mathbb{C})$ that $A + A^*$ is Hermitian and $A - A^*$ is skew-Hermitian.
   (b) Prove for all $A \in M_{n\times n}(\mathbb{C})$ that there exists a Hermitian matrix $B \in M_{n\times n}(\mathbb{C})$ and a skew-Hermitian matrix $C \in M_{n\times n}(\mathbb{C})$ such that

$$A = B + C.$$

   (c) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is skew-Hermitian then iA is Hermitian.
   (d) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is Hermitian and $k \in \mathbb{N}$ then A^k is Hermitian.
   (e) Prove that if $A \in M_{n\times n}(\mathbb{C})$ is skew-Hermitian and $k \in \mathbb{N}$ is even then A^k is Hermitian.
   (f) Prove that the eigenvalues of a skew-Hermitian matrix are pure imaginary.
   (g) Prove that the eigenvectors of a skew-Hermitian matrix are orthogonal.
   (h) Explain, given the properties proved above, why Hermitian matrices are analogous to real numbers, skew-Hermitian matrices are analogous to imaginary numbers, and generic matrices can be expressed in the form $A = \operatorname{Re}(A) + i\operatorname{Im}(A)$ for some Hermitian matrices $\operatorname{Re}(A)$ and $\operatorname{Im}(A)$.

5. Let $A \in M_{n\times n}(\mathbb{C})$ and $B = A^*A$.
   (a) Prove that B is a Hermitian matrix.
   (b) Prove that B is *positive semi-definite* meaning $\mathbf{v}^*B\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{C}^n$.
   (c) Prove that if A is full rank then B is *positive definite* meaning $\mathbf{v}^*B\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{C}^n$ and $\mathbf{v}^*B\mathbf{v} = 0$ if and only if $\mathbf{v} = 0$.

6. Let $A \in M_{n\times n}(\mathbb{C})$ with representation $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$ as described above. Prove that A is normal if and only if
$$\mathrm{Re}(A)\mathrm{Im}(A) = \mathrm{Im}(A)\mathrm{Re}(A).$$

7. Let $A \in M_{n\times n}(\mathbb{C})$.
   (a) If A is Hermitian, show that $\langle A\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in \mathbb{C}^n$.
   (b) If $\langle A\mathbf{v}, \mathbf{v}\rangle$ is real for all $\mathbf{v} \in \mathbb{C}^n$, prove that A is Hermitian. **Hint:** Consider the representation $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$.

8. Let $A \in M_{2\times 2}(\mathbb{C})$ be defined by
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$
   where $a, b \in \mathbb{C}$.
   (a) Show that the eigenvalues of A are $a \pm ib$.
   (b) If $a, b \in \mathbb{R}$, show that A is a scalar multiple of a unitary matrix.
   (c) Show that
$$A^*A = \begin{bmatrix} |a|^2 + |b|^2 & 2i\mathrm{Im}(ba^*) \\ -2i\mathrm{Im}(ba^*) & |a|^2 + |b|^2 \end{bmatrix}.$$
   (d) Show that A is a normal matrix.
   (e) Show that the singular values of A are given by
$$\sigma_1 = (|a|^2 + |b|^2 + 2|\mathrm{Im}(a^*b)|)^{\frac{1}{2}} \text{ and } \sigma_2 = (|a|^2 + |b|^2 - 2|\mathrm{Im}(a^*b)|)^{\frac{1}{2}}.$$