Lecture #13: Self-Adjoint Operators

Definition - A linear transformation $T: V \to V$ is self adjoint or Hermitian if for all $\vec{u}, \vec{v} \in V$:

$$\langle T(\vec{u}), \vec{v} \rangle = \langle \vec{u}, T(\vec{v}) \rangle.$$

Example:

Let $V = \{ f \in C^{\infty}([a,b]) : f(a) = f(b) = 0 \}$ and $T: V \to V$ be defined by

$$T(f) = \frac{d^2 f}{dx}.$$

Therefore, with respect to the standard inner product on function spaces we have that if $f, g \in V$ then

$$\begin{aligned}
\langle T(f), g \rangle &= \int_a^b \frac{d^2 f}{dx^2} g(x)\, dx \\
&= \cancel{\frac{df}{dx} g(x)}^{\,0} \Big|_a^b - \int_a^b \frac{df}{dx} \frac{dg}{dx}\, dx \\
&= -\int_a^b \frac{df}{dx} \frac{dg}{dx}\, dx \\
&= \cancel{-f \frac{dg}{dx}}^{\,0} \Big|_a^b + \int_a^b f(x) \frac{d^2 g}{dx^2}\, dx \\
&= \int_a^b f(x) \frac{d^2 g}{dx^2}\, dx \\
&= \int_a^b f(x)\, T(g(x))\, dx \\
&= \langle f, T(g) \rangle.
\end{aligned}$$

Therefore, T is self adjoint.

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Theorem - If $A \in M_{n\times n}(\mathbb{C})$ then for all $\vec{v}, \vec{w} \in \mathbb{C}^n$,

$$\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A^*\vec{w}\rangle$$

and A is self-adjoint if and only if $A = A^*$.

proof:

$$\langle A\vec{v}, \vec{w}\rangle = (A\vec{v})^*\vec{w} = \vec{v}^*A^*\vec{w} = \langle \vec{v}, A^*\vec{w}\rangle$$

which also proves that A is self-adjoint if and only if $A = A^*$.

Theorem - If A is Hermitian then all of its eigenvalues are real.

proof:

Let $\vec{v}$ be an eigenvector of A with corresponding eigenvalue λ.

Therefore,

$$\langle A\vec{v}, \vec{v}\rangle = \langle \vec{v}, A\vec{v}\rangle$$
$$\Rightarrow \langle \lambda\vec{v}, \vec{v}\rangle = \langle \vec{v}, \lambda\vec{v}\rangle$$
$$\Rightarrow \bar{\lambda}\langle \vec{v}, \vec{v}\rangle = \lambda\langle \vec{v}, \vec{v}\rangle$$
$$\Rightarrow \bar{\lambda} = \lambda$$
$$\Rightarrow \lambda \text{ is real}$$

Theorem - If A is Hermitian then eigenvectors of A corresponding to different eigenvalues are orthogonal.

proof:

Let $\vec{u}, \vec{v} \in V$ be eigenvectors of A with distinct eigenvalues $\lambda, \nu \in \mathbb{R}$

Consequently,

$$\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$$
$$\Rightarrow \langle \lambda\vec{u}, \vec{v}\rangle = \langle \vec{u}, \nu\vec{v}\rangle$$
$$\Rightarrow \lambda\langle \vec{u}, \vec{v}\rangle = \nu\langle \vec{u}, \vec{v}\rangle$$
$$\Rightarrow (\lambda - \nu)\langle \vec{u}, \vec{v}\rangle = 0$$

Since $\lambda \neq \nu$, $\langle \vec{u}, \vec{v}\rangle = 0$. ■

Theorem - Let A be a $n\times n$ Hermitian matrix. Then there exists unitary U and a diagonal matrix Δ with real entries such that

$$A = U^* \Delta U$$

Definition - If $A \in M_{m\times n}(\mathbb{C})$ then the $n\times n$ matrix A^*A is called the Gram matrix. A matrix A is called normal if $A^*A = AA^*$.

Theorem - Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A^*A with eigenvalues satisfying

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$$

Then $\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ is an orthogonal basis for $\operatorname{im}(A)$ and $\operatorname{rank}(A) = r$.

proof:

1. If $i \neq j$ we have

$$\langle A\vec{v}_i, A\vec{v}_j \rangle = \langle \vec{v}_i, A^*A\vec{v}_j \rangle = \lambda_j \langle \vec{v}_i, \vec{v}_j \rangle = 0.$$

2. Since $A\vec{v}_i = 0$ for $i > r$ and orthogonal vectors are linearly independent, it follows that $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ are linearly independent.

3. Let $\vec{u} \in \operatorname{Im}(A)$. Thereforefore, there exists $\vec{x} \in \mathbb{C}^n$ such that

$$\vec{u} = A\vec{x}$$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis it follows that there exists $c_1, \ldots, c_n \in \mathbb{C}$ such that

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

$$\Rightarrow \vec{u} = A\vec{x} = c_1 A\vec{v}_1 + \ldots + c_r A\vec{v}_r \Rightarrow \vec{u} \in \operatorname{span}\{A\vec{v}_1, \ldots, A\vec{v}_r\}. \quad \blacksquare$$