Lecture #14: Singular Value Decomposition

Theorem - Let $A \in M_{m\times n}(\mathbb{C})$ then there exists unitary matrices $U \in M_{m\times m}(\mathbb{C})$, $V \in M_{n\times n}(\mathbb{C})$ such that

$$A = U\Sigma V^*$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}.$$

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ and r is the rank of A. The quantities $\sigma_1, \ldots, \sigma_r$ are called the singular values of A.

proof:

Let $\{\vec{v}_1, \ldots, \vec{v}_r\}$ be an orthonormal basis of eigenvectors of A^*A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. Consequently, $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ is an ortogonal basis for $\mathrm{im}(A)$. Consequently, letting

$$\vec{u}_i = \frac{1}{\|A\vec{v}_i\|} A\vec{v}_i = \frac{1}{\sigma_i} A\vec{v}_i$$

it follows that $\{\vec{u}_1, \ldots, \vec{u}_r\}$ is an orthonormal basis for $\mathrm{im}(A)$. Extending to $\{\vec{u}_1, \ldots, \vec{u}_m\}$ we obtain an orthonormal basis for $\mathbb{C}^m$ and defining $U = [\vec{u}_1 | \cdots | \vec{u}_m]$, $V = [\vec{v}_1 | \cdots | \vec{v}_n]$ it follows that

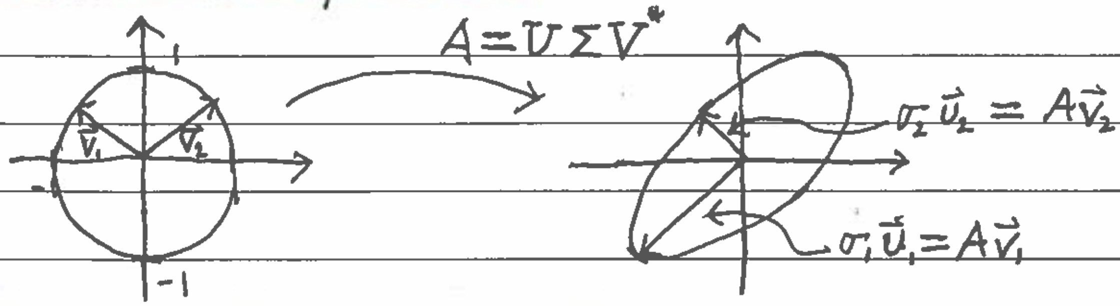
$$\boxed{\begin{aligned} &AV = U\Sigma \\ \Rightarrow\ &A = U\Sigma V^*. \end{aligned}}$$

Definition - The factorization

$$A = U\Sigma V^*$$

is called the singular value decomposition.

Geometric Intrepretation



$$\underbrace{[A]\left[\vec{v}_1 | \cdots | \vec{v}_n\right]}_{V} = \underbrace{\left[\vec{u}_1 | \cdots | \vec{u}_n\right]}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma}$$

unitary

Tips on finding the SVD

- $A^*A = V\Sigma U^* U \Sigma V^*$
  $= V\Sigma^2 V^*$

$\Rightarrow \vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of A^*A

$\Rightarrow \sigma_1^2, \ldots, \sigma_n^2$ are eigenvalues of A^*A

$\Rightarrow \sigma_1 \vec{u}_1 = A\vec{v}_1, \ldots, \sigma_n = A\vec{v}_n$.

Examples

1. $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $A\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\sigma_1 = 3$, $\sigma_2 = 2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $B=\begin{bmatrix}2&0\\0&3\end{bmatrix}$, $A\vec{e}_1=\begin{bmatrix}2\\0\end{bmatrix}=2\begin{bmatrix}1\\0\end{bmatrix}$, $A\vec{e}_2=\begin{bmatrix}0\\3\end{bmatrix}=3\begin{bmatrix}0\\1\end{bmatrix}$

$\Rightarrow \vec{v}_1=\begin{bmatrix}0\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{u}_1=\begin{bmatrix}0\\1\end{bmatrix}$, $\vec{u}_2=\begin{bmatrix}1\\0\end{bmatrix}$, $\sigma_1=3, \sigma_2=2$

$$\Rightarrow B=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

3. $C=\begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix}$, $A\vec{e}_1=0$, $A\vec{e}_2=\begin{bmatrix}2\\0\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\vec{v}_1=\begin{bmatrix}0\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{u}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{u}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{u}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\sigma_1=3, \sigma_2=0$

$$C=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{3\times3}\underbrace{\begin{bmatrix}3&0\\0&0\\0&0\end{bmatrix}}_{3\times2}\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{2\times2}$$

4. $D=\begin{bmatrix}1&1\\0&0\end{bmatrix} \Rightarrow \text{im}(A)=\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

We need to find direction of greatest stretching. Let
$v_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \Rightarrow D\vec{v}_1=\begin{bmatrix}2/\sqrt{2}\\0\end{bmatrix}=\sqrt{2}\cdot\begin{bmatrix}1\\0\end{bmatrix}$

Therefore,
$\vec{v}_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$, $\sigma_1=\sqrt{2}$, $\vec{u}_1=\begin{bmatrix}1\\0\end{bmatrix}$

Rank-nullity implies $\sigma_2=0$ and orthogonality implies
$\vec{v}_2=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$, $\sigma_2=0$, $\vec{u}_2=\begin{bmatrix}0\\1\end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^*$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

5. $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Lets find. $\ker(E)$. Row reducing we

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Consequently,

$$\ker(E) = \text{span}\left(\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right)$$

Therefore,

$$\vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \sigma_2 = 0$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Now,

$$E\vec{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Consequently, $\sigma_1 = 2$,

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

By orthogonality

$$\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow E = U \Sigma V^* = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Point of SVD

If $A = U \Sigma V^*$ then

$$A = \underbrace{\sigma_1 \vec{u}_1 \vec{v}_1^*} + \sigma_2 \vec{u}_2 \vec{v}_2^* + \dots + \sigma_n \vec{u}_n \vec{v}_n$$

: best rank-1 approximation
of a matrix.