Lecture #17: Square Roots and Exponentials

Definition - A is called positive semidefinite if A is Hermitian and for all $\vec{v} \in \mathbb{C}^n$:

$$\vec{v}^* A \vec{v} = \langle A\vec{v}, \vec{v} \rangle \geq 0$$

A is called positive definite if $\langle A\vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = 0$.

Theorem - If $A \in M_{n \times n}(\mathbb{C})$ is positive definite then the mapping $\langle \cdot, \cdot \rangle_A$ defined by

$$\langle \vec{x}, \vec{y} \rangle_A = \langle A\vec{x}, \vec{y} \rangle$$

is an inner product.

Theorem - Let A be positive semidefinite with diagonalization

$$A = U \Delta U^*$$

Then $\sqrt{A} = U \Delta^{1/2} U^*$

proof:

$$\begin{aligned}
\sqrt{A}\sqrt{A} &= U \Delta^{1/2} U^* U \Delta^{1/2} U^* \\
&= U \Delta^{1/2} \Delta^{1/2} U^* \\
&= U \Delta U^*
\end{aligned}$$

Definition - If $A \in M_{n \times n}(\mathbb{C})$ then

$$\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \ldots$$

Theorem - If A is diagonalizble in the form

$$A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$$

then

$$\exp(A) = U \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} U^*$$

proof:

$$\begin{aligned} \exp(A) &= I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \dots \\ &= I + U\Lambda U^* + \tfrac{1}{2}U\Lambda^2 U^* + \tfrac{1}{3!}U\Lambda^3 U^* + \dots \\ &= U(I + \Lambda + \tfrac{1}{2}\Lambda^2 + \tfrac{1}{3!}\Lambda^3 + \dots)U^* \\ &= U \begin{bmatrix} 1+\lambda_1+\tfrac{1}{2}\lambda_1^2+\dots & & 0 \\ & \ddots & \\ 0 & & 1+\lambda_n+\tfrac{1}{2}\lambda_n^2+\dots \end{bmatrix} U^* \\ &= U \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} U^* \end{aligned}$$

Example:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$Tr(A) = 4 = \lambda_1 + \lambda_2$

$det(A) = 5 = \lambda_1 \lambda_2$

$\Rightarrow \lambda_1 = 4 - \lambda_2$

$5 = (4-\lambda_2)\lambda_2$

$\Rightarrow \lambda_2^2 - 4\lambda_2 + 5 = 0$

$\lambda_2 = 2 \pm i$

The eigenvectors are

$$\vec{V}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Consequently,

$$\exp(A) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{2+i} & 0 \\ 0 & e^{2-i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}$$

$$= \frac{e^2}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{e^2}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} \cos(1) + i\sin(1) & 0 \\ 0 & \cos(1) - i\sin(1) \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \begin{bmatrix} e^2\cos(1) & -e^2\sin(1) \\ e^2\sin(1) & e^2\cos(1) \end{bmatrix}$$

<u>Polar Form</u> - Let $A \in M_{n\times n}(\mathbb{C})$. There exists unitary matrices $R, Q \in M_{n\times n}(\mathbb{C})$ such that

$$A = Q\sqrt{A^*A}$$

$$A = \sqrt{AA^*}R$$

proof:

Let A have the SVD, $A = U\Sigma V^*$. Therefore,

$$A^*A = V\Sigma^2V^* \Rightarrow \sqrt{A^*A} = V\Sigma V^*$$

$$AA^* = U\Sigma^2U^* \Rightarrow \sqrt{AA^*} = U\Sigma U^*$$

Now,

$$A = U\Sigma V^* = \underbrace{UV^*}_{Q}\underbrace{V\Sigma V^*}_{\sqrt{A^*A}}$$

$$A = U\Sigma V^* = \underbrace{U\Sigma U^*}_{\sqrt{AA^*}}\underbrace{UV^*}_{R}$$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find SVD!

$\ker(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \sigma_2 = 0$

$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, A\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Therefore, $\sigma_1 = 2$, $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Finally, by orthonormality $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Consequently,

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow A = U \Sigma V^*$$
$$= U V^* V \Sigma V^*$$

$$U V^* = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V \Sigma V^* = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A = Q\sqrt{A^*A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Theorem - If U is unitary, then $U = e^{iH}$ for some Hermitian matrix H.

proof:

Since U is unitarily diagonalizable with eigenvalues $|\lambda_j| = 1$ there exists $\theta_j \in \mathbb{R}$ such that $\lambda_j = e^{i\theta_j}$. Therefore,

$$U = R\Delta R^*$$
$$= R\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} R^*$$
$$= R\begin{bmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{bmatrix} R^*$$
$$= \exp\left(iR\begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} R^* \right)$$
$$\Rightarrow H = R\begin{bmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_n \end{bmatrix} R^* \leftarrow \text{Hermitian matrix.}$$

Theorem - Let $A \in M_{n\times n}(\mathbb{C})$, then there exists Hermitian matrices H_1, H_2 such that

$$A = e^{iH_1}\sqrt{A^*A} = \sqrt{AA^*}e^{iH_2}.$$

proof

Follows from polar decomposition.

Example:

$$A=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

$\ker(A)=\operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$\Rightarrow \vec{v}_2=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix},\ \sigma_2=0$

$\Rightarrow \vec{v}_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix},\quad A\vec{v}_1=\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}2/\sqrt{2}\\0\end{bmatrix}=2/\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix}=\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix}.$

Therefore,

$\vec{u}_1=\begin{bmatrix}1\\0\end{bmatrix},\ \sigma_1=\sqrt{2}$ and thus by orthogonality $\vec{u}_2=\begin{bmatrix}0\\1\end{bmatrix}.$

Consequently,

$$U=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\quad V=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix},\quad \Sigma=\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}$$

Now,

$$\begin{aligned}A&=U\Sigma V^*\\&=UV^*V\Sigma V^*\\&=Q\sqrt{A^*A}\end{aligned}$$

$$\begin{aligned}Q&=UV^*\\&=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\end{aligned}$$

$\operatorname{Tr}(Q)=\lambda_1+\lambda_2=0$

$\operatorname{Det}(Q)=-1=\lambda_1\cdot\lambda_2$

$\Rightarrow \lambda_1=1,\ \lambda_2=-1$

$\underline{\lambda_1=1}$

$$Q-I=\begin{bmatrix}(1-\sqrt{2})/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & (-1-\sqrt{2})/\sqrt{2}\end{bmatrix}$$

$\Rightarrow$ The eigenvector is $\vec{w}_1=\begin{bmatrix}1\\1+\sqrt{2}\end{bmatrix}$

$\underline{\lambda_2=-1}$

$$Q+I=\begin{bmatrix}(1+\sqrt{2})/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & (-1+\sqrt{2})/\sqrt{2}\end{bmatrix}$$

$\Rightarrow$ The eigenvector is

$\vec{w}_2=\begin{bmatrix}1\\1-\sqrt{2}\end{bmatrix}$

Consequently

$$Q = \begin{bmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/4(2-\sqrt{2}) & 1/(2\sqrt{2}) \\ 1/4(2+\sqrt{2}) & -1/(2\sqrt{2}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^{i\pi} \end{bmatrix} \begin{bmatrix} 1/4(2-\sqrt{2}) & 1/(2\sqrt{2}) \\ 1/4(2+\sqrt{2}) & -1/(2\sqrt{2}) \end{bmatrix}$$

$$= \exp\left(i \begin{bmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \pi \end{bmatrix} \begin{bmatrix} 1/4(2-\sqrt{2}) & 1/(2\sqrt{2}) \\ 1/4(2+\sqrt{2}) & -1/(2\sqrt{2}) \end{bmatrix} \right)$$

$$= \exp\left(\frac{i}{4} \begin{bmatrix} (2+\sqrt{2})\pi & -\sqrt{2}\pi \\ -\sqrt{2}\pi & (2-\sqrt{2})\pi \end{bmatrix} \right)$$

Furthermore,

$$A^*A = V \Sigma^2 V^*$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \sqrt{A^*A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Consequently,

$$A = e^{iH} \sqrt{A^*A}$$

$$= \exp\left(\frac{i}{4} \begin{bmatrix} (2+\sqrt{2})\pi & -\sqrt{2}\pi \\ -\sqrt{2}\pi & (2-\sqrt{2})\pi \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$