MTH 225
Quiz #9

1. Let $A \in M_{3\times 3}(\mathbb{C})$ be given by

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Compute e^A.

Method #1:

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I$$

$$\Rightarrow A^3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -A$$

$$A^4 = I$$

Therefore,

$$e^A = I + A - \frac{1}{2}I - \frac{1}{3!}A + \frac{1}{4!}I + \frac{1}{5!}A + \dots$$

$$= \left(1 - \frac{1}{2} + \frac{1}{4!} + \dots\right)I + \left(1 - \frac{1}{3!} + \frac{1}{5!} + \dots\right)A$$

$$= \cos(1)I + \sin(1)A$$

$$= \begin{bmatrix} \cos(1) & i\sin(1) \\ i\sin(1) & \cos(1) \end{bmatrix}$$

Method #2:

$Tr(A) = 0$, $\det(A) = 1$

Therefore, the eigenvalues are

$$\lambda = \pm i$$

$\lambda = i$:

$$\ker(\lambda I - A) = \ker\left(\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}\right)$$

$$\Rightarrow \ker(\lambda I - A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$\lambda = -i$:

$$\ker(\lambda I - A) = \ker\left(\begin{bmatrix} -i & -i \\ -i & -i \end{bmatrix}\right)$$

$$\Rightarrow \ker(\lambda I - A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$\Rightarrow e^A = \frac{-1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^i & 0 \\ 0 & e^{-i} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -e^i & -e^i \\ -e^{-i} & e^{-i} \end{bmatrix}$$

$$= \frac{-1}{2}\begin{bmatrix} -e^i - e^{-i} & -e^i + e^{-i} \\ -e^i + e^{-i} & -e^i - e^{-i} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(1) & i\sin(1) \\ i\sin(1) & \cos(1) \end{bmatrix}$$