MTH 352/652
Spring 2025
Exam 2
04/02/25

Name (Print): Key

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

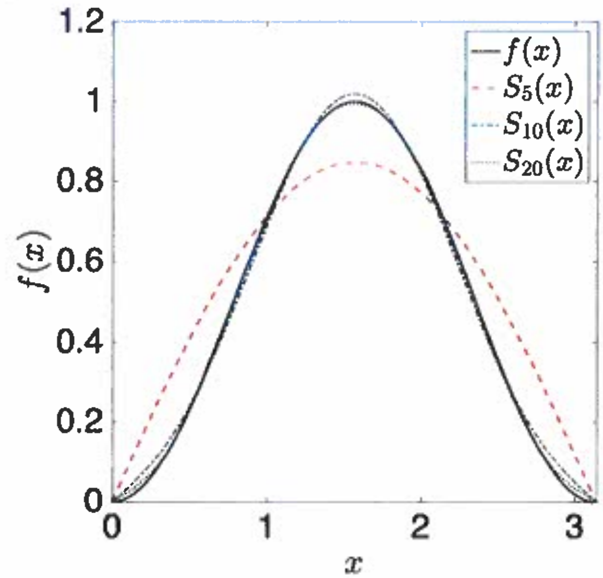
The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

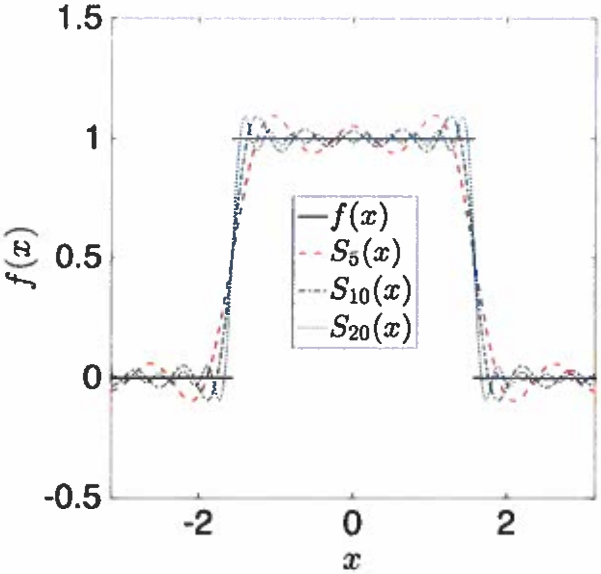
Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	10	
6	15	
7	15	
8	15	
Total:	100	

1. (10 points) **Short Answer:** In the figures below a function $f(x) \in L^2[a,b]$ is plotted along with the Fourier series approximations $S_n(x) = \sum_{i=1}^{n} a_i h_i(x)$, where h_i is some complete orthonormal system on the interval $[a,b]$ and $a_i \in \mathbb{R}$. For each figure, indicate whether the partial sums S_n converge in L^2, uniformly, and/or pointwise to $f(x)$ by checking the appropriate box. **Note:** The series could converge in more than one sense.

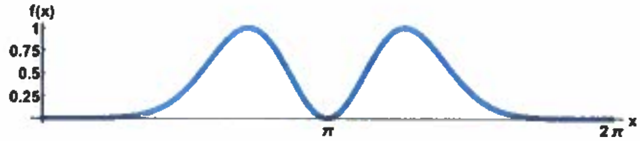


- [x] Converges in L^2 norm
- [x] Converges pointwise
- [x] Converges uniformly



- [x] Converges in L^2 norm
- [ ] Converges pointwise
- [ ] Converges uniformly

2. (10 points) **Short Answer:** Suppose $f(x)$ is the function plotted below:



For each of the below initial-boundary value problems defined for $x \in [0, 2\pi]$ and $t > 0$, match them with one of the corresponding contour plots of the solution $u(x,t)$. Note, one of the plots will not be used.

(*i*) (c)

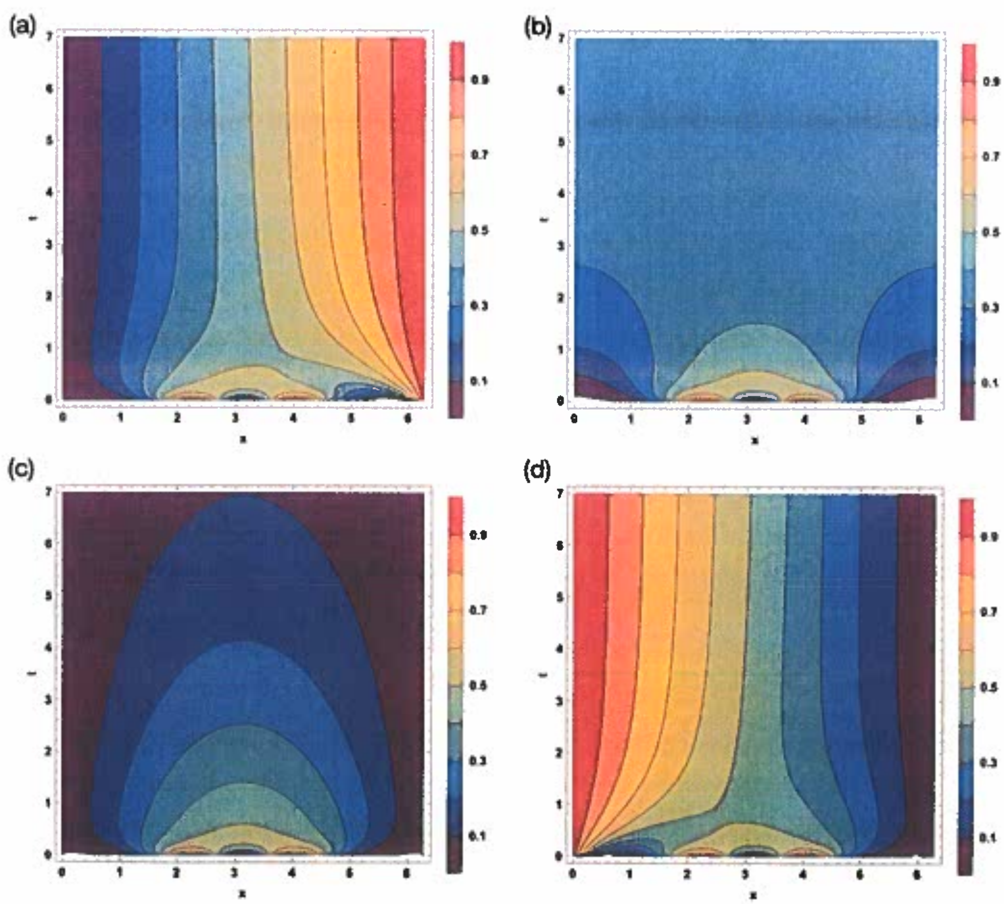
$$u_t = u_{xx},\quad u(0,t) = 0,\quad u(2\pi,t) = 0,\quad u(x,0) = f(x),$$

(*ii*) (b)

$$u_t = u_{xx},\quad u_x(0,t) = 0,\quad u_x(2\pi,t) = 0,\quad u(x,0) = f(x),$$

(*iii*) (a)

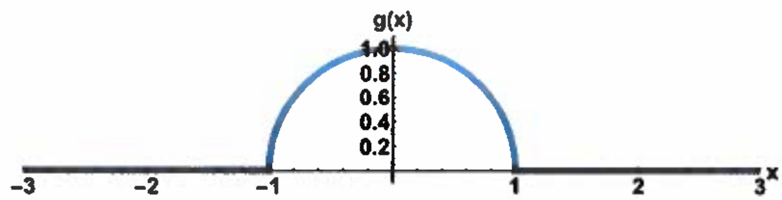
$$u_t = u_{xx},\quad u(0,t) = 0,\quad u(2\pi,t) = 1,\quad u(x,0) = f(x).$$



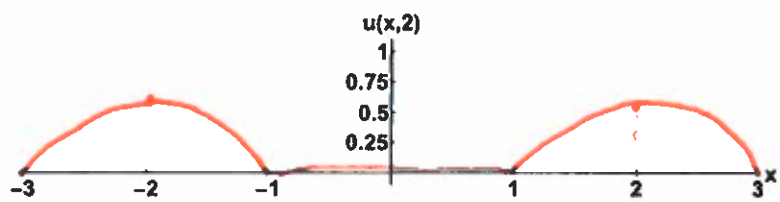
3. (15 points) Suppose $u(x,t)$ is a solution to the wave equation on $\mathbb{R}$:

$$u_{tt} = u_{xx},$$
$$u(x,0) = g(x),$$
$$u_t(x,0) = 0,$$

where $g(x)$ is the half circular pulse of radius 1 plotted below.



(a) (5 points) **Short Answer:** On the figure below, sketch a graph of $u(x,2)$.



(b) (5 points) **Short Answer:** Compute

$$u^*(x) = \lim_{t\to\infty} u(x,t).$$

0

(c) (5 points) **Short Answer:** Compute

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} u(x,t)dx.$$

$\pi/2$

4. (10 points) Consider the following initial value problem for $x \in \mathbb{R}$ and $t \geq 0$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial t} - 20\frac{\partial^2}{\partial t^2}\right) u = 0,$$
$$u(x,0) = \phi(x),$$
$$u_t(x,0) = 0,$$

where $\phi(x)$ is a Schwartz class function.

(a) (5 points) Factor the above equation into the form

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x} + d\frac{\partial}{\partial t}\right) u = 0,$$

for some $a, b, c, d \in \mathbb{R}$.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x \partial t} - 20\frac{\partial^2}{\partial t^2}\right) = \left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)$$

(b) (5 points) **Short Answer:** Using your factorization, right down the solution to this initial value problem in terms of ϕ. **You do not have to justify anything, you can just write down the solution.**

$u(x,t) = a\,\phi(x - \tfrac{1}{5}t) + b\,\phi(x + \tfrac{1}{4}t)$

$u(x,0) = \phi(x)$

$\Rightarrow a + b = 1$

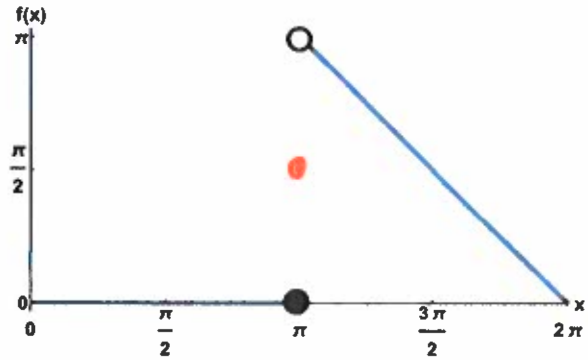
$u_t(x,0) = 0$

$\Rightarrow -\tfrac{1}{5}a + \tfrac{1}{4}b = 0$

$\Rightarrow -4a + 5b = 0$

$\Rightarrow a = 5/9,\ b = 4/9$

5. (10 points) Suppose $f(x)$ is the function plotted below on the domain $[0, 2\pi]$:



Given the orthogonal system $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$, the Fourier series representation of $f(x)$ on $[0, 2\pi]$ is:

$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx).$$

The partial sums are defined by

$$S_N(x) = \frac{\pi}{4} + \sum_{n=1}^{N} \frac{(-1)^n - 1}{\pi n^2} \cos(nx) + \sum_{n=1}^{N} \frac{(-1)^n}{n} \sin(nx).$$

(a) (5 points) **Short Answer:** Compute

$$\lim_{N \to \infty} S_N(\pi).$$

$$\frac{\pi}{2}$$

(b) **Short Answer:** Compute

$$\lim_{N \to \infty} \int_0^{2\pi} (f(x) - S_N(x))^2 \, dx.$$

$$0$$

6. (15 points) Using the orthogonal system $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ on the interval $[-\pi. \pi]$, calculate the Fourier series representation of $f(x) = x$ on the interval $[-\pi, \pi]$.

Since x is odd, only the sin terms are relevant.

$$\Rightarrow x = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \int_{-\pi}^{\pi} x \sin(nx)\,dx = \int_{-\pi}^{\pi} b_n \sin^2(nx)\,dx$$

$$\Rightarrow 2\int_{0}^{\pi} x \sin(nx)\,dx = b_n \pi$$

$$\Rightarrow \frac{2}{\pi}\int_{0}^{\pi} x \sin(nx)\,dx = b_n$$

$$\Rightarrow b_n = \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n}\Big|_0^{\pi} + \underbrace{\int_0^{\pi} \frac{1}{n}\cos(nx)\,dx}_{0}\right)$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\Rightarrow x \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx).$$

7. (15 points) For $L > 0$, the following partial differential equation models the heat flow in a pipe of length L:

$$
\begin{aligned}
u_t &= u_{xx}, \\
u(0,t) &= 0, \\
u_x(L,t) &= 0.
\end{aligned}
$$

Note, I did not provide any initial conditions for this problem as it is not relevant to parts (a) and (b) below.

(a) (5 points) **Short Answer:** Briefly interpret what the boundary conditions tell you about the heat at the boundary of the domain.

The heat is held constant at $x=0$ and there is no flux of heat at $x=L$.

(b) (10 points) By assuming a solution of the form $u(x,t) = X(x)T(t)$, find all separable solutions to this boundary value problem.

$$U_t = U_{xx}$$

$$\Rightarrow XT' = X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\Rightarrow X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),\ T = Ce^{-\lambda t}$$

Boundar conditions imply

$$X(0) = A = 0$$

$$X'(L) = -B\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \sqrt{\lambda}L = \frac{(2n+1)\pi}{2}$$

Therefore,

$$XT = 1 \quad \text{or} \quad XT = e^{-(2n+1)^2\pi^2/4L^2 t}\cos\left(\frac{(2n+1)\pi}{2L}x\right)$$

8. (15 points) Consider the space of functions

$$V = \{f \in L^2([0,1]) : f'(0) = f(0) \text{ and } f'(L) = -f(L)\}$$

and let $\langle \cdot, \cdot \rangle$ be the standard inner product on V defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

(a) (5 points) **Short Answer:** Write down the definition of what it means for an operator $\mathcal{L}$ to be self adjoint on V with respect to this inner product.

$\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}g \rangle$

(b) (10 points) Prove that the operator $\mathcal{L}$ defined by

$$\mathcal{L}f = \frac{d^2 f}{dx^2},$$

is self adjoint on V with respect to this inner product.

$$\langle \mathcal{L}f, g \rangle = \int_0^1 f''(x)g(x)dx$$
$$= f'(x)g(x)\Big|_0^1 - \int_0^1 f'(x)g'(x)dx$$
$$= -f(1)g(1) - f(0)g(0) - f(x)g'(x)\Big|_0^1 + \int_0^1 f(x)g''(x)dx$$
$$= -f(1)g(1) - f(0)g(0) + f(1)g(1) + f(0)g(0) + \langle f, \mathcal{L}g \rangle$$
$$= \langle f, \mathcal{L}g \rangle.$$