MTH 352/652
Homework #10

Due Date: April 25, 2025

1. Solve the following boundary value problem on a quarter wedge of radius R:

$$\begin{aligned}&\Delta u = 0,\ \Omega = \{(r,\theta) : r < R,\ 0 \le \theta \le \pi/2\},\\ &u(R,\theta) = \sin(2\theta),\\ &u(r,0) = u(r,\pi/2) = 0.\end{aligned}$$

2. Solve the following boundary value problem on an annulus:

$$\begin{aligned}&\Delta u = 0,\ \Omega = \{(r,\theta) : 1 < r < 2,\ 0 \le \theta \le 2\pi\},\\ &u(1,\theta) = 0,\\ &u(2,\theta) = \sin^2(\theta).\end{aligned}$$

3. Consider the wave equation on the *real line* $x \in \mathbb{R}$, $t > 0$ with a source term:

$$\begin{aligned}u_{tt} &= c^2 u_{xx} + \sin(x),\\ u(x,0) &= 0,\\ u_t(x,0) &= 0.\end{aligned}$$

   (a) Find all steady state solutions to this problem.

   (b) Solve this initial value problem.

4. Using Duhamel's principle, find a formula for the solution to the following initial value problem on the real line $x \in \mathbb{R}$, $t > 0$:

$$\begin{aligned}u_t + cu_x &= f(x,t),\\ u(x,0) &= 0.\end{aligned}$$

5. Solve the following initial value problem on the real line $x \in \mathbb{R}$, $t > 0$:

$$\begin{aligned}u_t + 2u_x &= xe^{-t},\\ u(x,0) &= 0.\end{aligned}$$

6. Consider the following initial boundary value problem on the domain $x \in [0, 1], t > -0$:

$$\begin{aligned} u_t &= k u_{xx}, \\ u(0, t) &= \sin(t), \\ u(1, t) &= 1, \\ u(x, 0) &= \sin(\pi x) + x. \end{aligned}$$

   (a) Give a physical interpretation of the boundary conditions for this problem.

   (b) Transform this problem into a problem with homogeneous boundary conditions.

   (c) Solve this initial-boundary value problem.

7. Solve the following initial boundary value problem for $x \in [0, 1]$, $t > 0$:

$$\begin{aligned} u_t &= u_{xx} + \sin(3\pi x), \\ u(0, t) &= 0, \\ u(1, t) &= 0, \\ u(x, 0) &= \sin(\pi x). \end{aligned}$$

8. Consider the following wave equation with a constant gravitational force g modeling the dynamics of rope of length L:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - g, \\ u(0, t) &= 0, \\ u(L, t) &= \sin(t), \\ u(x, 0) &= x(L - x), \\ u_t(x, 0) &= 0. \end{aligned}$$

Homework #10

#1

Solve the following boundary value problem

$$\Delta u = 0$$
$$u(R,\theta) = \sin(2\theta)$$
$$u(r,0) = u(r,\pi/2) = 0$$

Solution:

From the boundary conditions the solution is of the following form:

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin(2n\theta).$$

Since $u(R,\theta) = \sin(2\theta)$ it follows that

$$u(r,\theta) = \frac{r^2}{R^2} \sin(2\theta).$$

#2

Solve the following boundary value problem

$$\Delta u = 0,$$
$$u(1,\theta) = 0$$
$$u(2,\theta) = \sin^2\theta = (1-\cos(2\theta))/2.$$

Solution:

The generic form of the seperable solutions is

$$u_n(r,\theta) = R_n(r)\,\Theta_n(\theta)$$

where

$$R_n(r) = \begin{cases} c_n r^n + d_n r^{-n}, & n \neq 0 \\ c_0 + d_0 \ln(r), & n = 0 \end{cases}, \quad \Theta_n(\theta) = \begin{cases} a_n \cos(n\theta) + b_n \sin(n\theta), & n \neq 0 \\ a_0 & , n = 0 \end{cases}$$

Since $u(1,\theta)=0$ it follows that $R_n(1)=0$ and thus

$$\begin{cases} c_n + d_n = 0, & n \neq 0 \\ c_0 = 0, & n = 0 \end{cases}$$

Therefore, absorbing constants we have that

$$u(r,\theta) = a_0 \ln(r) + \sum_{n=1}^{\infty} (r^n - r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta))$$

Consequently,

$$\frac{1}{2} - \frac{\cos(2\theta)}{2} = a_0 \ln(r) + \sum_{n=1}^{\infty} (2^n + 2^{-n})(a_n \cos(2\theta) + b_n \sin(n\theta))$$

$\Rightarrow a_0 = \frac{1}{2\ln(2)}$ and $-\frac{1}{2} = (4 - \frac{1}{4}) a_2$ all other coefficients are zero.

Therefore, $a_0 = \frac{1}{2\ln(2)}$, $a_2 = -\frac{2}{15}$ and thus:

$$u(r,\theta) = \frac{\ln(r)}{2\ln(2)} - \frac{2(r^2 - r^{-2})}{15} \cos(2\theta).$$

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#3

Consider the wave equation on the real line $x \in \mathbb{R}$, $t > 0$ with a source term:

$$u_{tt} = c^2 u_{xx} + \sin(x)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

(a) Find all steady state solutions to this problem

(b) Solve this initial value problem.

Solution:

(a) Steady state solutions satisfy:

$$U^*_{xx} = 1/c^2 \sin(x)$$

$$\Rightarrow U^*(x) = -1/c^2 \sin(x) + ax + b.$$

(b) Letting $V = U - U^*$ it follows that

$$V_{tt} = c^2 V_{xx}$$
$$V(x,0) = 1/c^2 \sin(x) - ax - b$$
$$V_t(x,0) = 0$$

Therefore,

$$V(x,t) = \frac{1}{2}\left(1/c^2 \sin(x-ct) - a(x-ct) - b + \frac{1}{c^2}\sin(x+ct) - a(x+ct) - b \right).$$

Therefore,

$$u(x,t) = \frac{1}{2c^2}(\sin(x-ct) + \sin(x+ct)) - \frac{a}{2}(x-ct+x+ct)$$
$$+ \frac{1}{2}(-b-b) - \frac{1}{c^2}\sin(x) + ax + b.$$

Therefore,

$$u(x,t) = \frac{1}{2c^2}(\sin(x-ct) + \sin(x+ct)) - \frac{1}{c^2}\sin(x).$$

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#4

Using Duhamel's principle, find a formula for the solution to the following initial value problem on the real line $x \in \mathbb{R}$, $t > 0$:

$$U_t + cU_x = f(x,t)$$
$$U(x,0) = 0.$$

Solution:

Auxiliary problem:

$$w_t + c w_x = 0$$
$$w(x,0;\tau) = f(x,\tau)$$
$$\Rightarrow w(x,t;\tau) = f(x-ct,\tau)$$

Therefore,

$$u(x,t) = \int_0^t w(x,t-\tau;\tau)\,d\tau$$
$$= \int_0^t f(x-c\tau,\tau)\,d\tau.$$

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#5.

Solve the following initial value problem on the real line $x \in \mathbb{R}$, $t>0$:

$$u_t + 2u_x = xe^{-t}$$
$$u(x,0) = 0$$

Solution:

Auxiliary problem

$$w_t + 2u_x = 0$$
$$w(x,t;\tau) = xe^{-\tau}$$
$$\Rightarrow w(x,t;\tau) = (x-2t)e^{-\tau}.$$

Therefore,

$$u(x,t) = \int_0^t w(x,t-\tau;\tau)\,d\tau$$
$$= \int_0^t (x-2(t-\tau))e^{-\tau}\,d\tau$$
$$= \int_0^t (xe^{-\tau} - 2te^{-\tau} + 2\tau e^{-\tau})\,d\tau$$
$$= (1-e^{-t})x - 2t(1-e^{-t}) - 2te^{-t} + 2(1-e^{-t})$$
$$= (1-e^{-t})(x-2t+2) - 2te^{-t}.$$

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#6

Consider the following boundary value problem:

$$U_t = k U_{xx}$$
$$u(0,t) = \sin(t)$$
$$u(1,t) = 1$$
$$u(x,0) = \sin(\pi x) + x$$

(a) Give a physical interpretation of the boundary conditions.

(b) Transform this problem into a problem with homogeneous boundary conditions.

(c) Solve this initial-boundary problem.

Solution:

(a) This could model the flow of heat in a unit length rod. The rod is held at constant temperature at its left endpoint while the right endpoint temperature fluctuates periodically in time.

(b) Assume $v = u - A(t) - B(t)x$. If we assume

$$v(0,t) = v(1,t) = 0$$
$$\Rightarrow 0 = u(0,t) - A(t), \quad 0 = u(1,t) - \sin(t) - B(t)$$
$$\Rightarrow 0 = \sin(t) - A(t), \quad 0 = 1 - \sin(t) - B(t)$$
$$\Rightarrow A(t) = \sin(t) \qquad B(t) = 1 - \sin(t).$$

Now,

$$v_t = u_t - A'(t) - B'(t)x = k u_{xx} - \cos(t) + \cos(t)x$$
$$v_{xx} = u_{xx}$$
$$v(x,0) = u(x,0) - A(0) - B(0)x = \sin(\pi x) + x - x = \sin(\pi x).$$

Therefore,

$$v_t = K v_{xx} - \cos(t) + \cos(t) x$$
$$v(0,t) = 0$$
$$v(1,t) = 0$$
$$v(x,0) = \sin(\pi x)$$

<u>Problem #1</u>

$$\bar{v}_t = K v_{xx}$$
$$\bar{v}(0,t) = 0$$
$$\bar{v}(1,t) = 0$$
$$\bar{v}(x,0) = \sin(\pi x)$$

$$\Rightarrow \bar{v}(x,t) = e^{-K\pi^2 t} \sin(\pi x)$$

<u>Problem #2</u>

$$\tilde{v}_t = K \tilde{v}_{xx} + \cos(t)(x-1)$$
$$\tilde{v}(0,t) = 0$$
$$\tilde{v}(1,t) = 0$$
$$\tilde{v}(x,0) = 0$$

<u>Auxiliary Problem:</u>

$$w_t = K w_{xx}$$
$$w(0,t;\tau) = 0$$
$$w(1,t;\tau) = 0$$
$$w(x,0;\tau) = \cos(\tau)(x-1)$$

Therefore,

$$w(x,t;\tau)=\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt}\sin(n\pi x)$$

where

$$b_n = 2\int_0^1 \cos(\tau)(x-1)\sin(n\pi x)\,dx$$

$$=2\cos(\tau)\left(-\frac{(x-1)}{n\pi}\cos(n\pi x)\Big|_0^1+\frac{1}{n\pi}\int_0^1\cos(n\pi x)\,dx\right)$$

$$=-\frac{2\cos(\tau)}{n\pi}$$

Therefore,

$$w(x,t;\tau)=-\frac{2\cos(\tau)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2 kt}\sin(n\pi x)$$

$$\Rightarrow \tilde{v}(x,t)=\int_0^t w(x,t-\tau;\tau)\,d\tau$$

$$=\int_0^t -\frac{2\cos(\tau)}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2 k(t-\tau)}\sin(n\pi x)\,d\tau$$

$$=\sum_{n=1}^{\infty}\frac{2}{\pi n}\left(\frac{kn^2\pi^2(e^{-kn^2\pi^2 t}-\cos(t))-\sin(t)}{1+k^2n^4\pi^4}\right)\sin(n\pi x)$$

Therefore,

$$u(x,t)=v(x,t)+\sin(t)+(1-\sin(t))x$$

$$=\hat{v}(x,t)+\tilde{v}(x,t)+(1-\sin(t))x$$

$$\Rightarrow \boxed{u(x,t)=e^{-k\pi^2 t}\sin(\pi x)+(1-\sin(t))x+\sum_{n=1}^{\infty}\frac{2}{\pi n}\left(\frac{kn^2\pi^2(e^{-kn^2\pi^2 t}-\cos(t))-\sin(t)}{1+k^2n^4\pi^4}\right)\sin(n\pi x)}$$

$\blacksquare$

#7

Solve the following initial value problem

$$U_t = U_{xx} + \sin(3\pi x)$$
$$u(0,t)=0$$
$$u(1,t)=0$$
$$u(x,0)=\sin(\pi x)$$

Solution:

The steady state solution is

$$U^*(x) = \frac{1}{9\pi^2}\sin(3\pi x)$$

If we let $v = u - u^*(x)$ we have that

$$V_t = V_{xx}$$
$$v(0,t)=0$$
$$v(1,t)=0$$
$$v(x,0)=\sin(\pi x) - \frac{1}{9\pi^2}\sin(3\pi x)$$

$$\Rightarrow v(x,t) = e^{-\pi^2 t}\sin(\pi x) - \frac{1}{9\pi^2}e^{-9\pi^2 t}\sin(3\pi x)$$

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x) - \frac{1}{9\pi^2}e^{-9\pi^2 t}\sin(3\pi x) + \frac{1}{9\pi^2}\sin(3\pi x)$$

#8

Solve the following wave equation with a constant gravitational force g modelling the dynamics of a rope of length L:

$$U_{tt} = c^2 U_{xx} - g$$
$$u(0,t) = 0$$
$$u(L,t) = \sin(t)$$
$$U(x,0) = x(L-x)$$
$$U_t(x,0) = 0$$

<u>Solution:</u>

Let $v(x,t) = u - A(t) - B(t)x$. Therefore,

$$v(0,t) = u(0,t) - A(t),$$
$$= -A(t),$$

Therefore, we set $A(t) = 0$ so that $v(0,t) = 0$. Similarly,

$$v(L,t) = u(L,t) - B(t)L$$
$$= \sin(t) - B(t)L$$

and thus we set

$$B(t) = \sin(t)/L.$$

Consequently,

$$V_{tt} = U_{tt} - \frac{1}{L}\sin(t)$$
$$= c^2 U_{xx} - g - \frac{1}{L}\sin(t)$$

$$\Rightarrow V_{tt} = c^2 V_{xx} - g - \frac{1}{L}\sin(t)x$$
$$V(0,t) = 0$$
$$V(L,t) = 0$$
$$V(x,0) = x(L-x)$$
$$V_t(x,0) = -\frac{1}{L}x$$

Problem #1

$$\bar{v}_{tt} = c^2 \bar{v}_{xx}$$
$$\bar{v}(0,t) = 0$$
$$\bar{v}(L,t) = 0$$
$$\bar{v}(x,0) = x(L-x)$$
$$\bar{v}_t(x,0) = 0$$

Boundary and initial conditions imply that

$$\bar{v}(x,t) = \sum_{n=1}^{\infty} b_n \cos(cn\pi t/L)\sin(n\pi x/L)$$

$$\Rightarrow b_n = 2/L \int_0^L x(L-x)\sin(n\pi x/L)\,dx$$
$$= \frac{4(1-(-1)^n)L^2}{n^3\pi^3}.$$

Problem #2

$$\tilde{v}_{tt} = c^2 \tilde{v}_{xx}$$
$$\tilde{v}(0,t) = 0$$
$$\tilde{v}(L,t) = 0$$
$$\tilde{v}(x,0) = 0$$
$$\tilde{v}_t(x,0) = -1/L\, x$$

Boundary and initial conditions imply that

$$\tilde{v}(x,t) = \sum_{n=1}^{\infty} c_n \sin(cn\pi t/L)\sin(n\pi x/L)$$
$$\tilde{v}_t(x,t) = \sum_{n=1}^{\infty} c_n \cdot cn/L \cos(cn\pi t/L)\sin(n\pi x/L)$$

$$\Rightarrow c_n = \frac{2}{cn\pi}\int_0^L \sin(n\pi x/L)(-1/L\, x)\,dx$$
$$= \frac{2(-1)^n L}{cn^2\pi^2}$$

Problem #3

$\hat{v}_{tt} = c^2 \hat{v}_{xx} - g - 1/L \sin(t) x$

$\hat{v}(0,t) = 0$

$\hat{v}(L,t) = 0$

$\hat{v}(x,0) = 0$

$\hat{v}_t(x,0) = 0.$

Auxilliary Problem:

$w_{tt} = c^2 w_{xx}$

$w(0,t;\tau) = 0$

$w(L,t;\tau) = 0$

$w(x,0;\tau) = 0$

$w_t(x,0;\tau) = -g - 1/L \sin(\tau) x$

Initial and Boundary conditions imply that

$$w(x,t;\tau) = \sum_{n=1}^{\infty} d_n \sin(n\pi c t/L) \sin(n\pi x/L)$$

$$\Rightarrow w_t(x,0;\tau) = \sum_{n=1}^{\infty} d_n n\pi c/L \sin(n\pi x/L)$$

$$\Rightarrow d_n = \frac{2}{n\pi c} \int_0^L \sin(n\pi x/L)(-g - 1/L \sin(\tau) x) dx$$

$$= \frac{-4}{c n^2 \pi^2} (gL + \sin(\tau)) \sin\left(\frac{n\pi}{2}\right)^2$$

Consequently,

$$\hat{v}(x,t) = \int_0^t w(x, t-\tau; \tau) d\tau$$

$$= \int_0^t \sum_{n=1}^{\infty} \frac{-4}{c n^2 \pi^2} (gL + \sin(\tau)) \sin\left(\frac{n\pi}{2}\right)^2 \sin\left(\frac{n\pi c(t-\tau)}{L}\right) \sin\left(\frac{n\pi x}{L}\right) d$$

The full solution is then given by

$$u(x,t) = \bar{v}(x,t) + \tilde{v}(x,t) + \hat{v}(x,t) + \frac{1}{L}\sin(t)x.$$