MTH 352/652
Homework #9

Due Date: April 11, 2025

1. Consider the following initial-boundary value problem:

$$
\begin{aligned}
&u_t = u_{xx},\\
&u(0,t) = 1,\\
&u(2\pi,t) = 2,\\
&u(x,0) = 1.
\end{aligned}
$$

   (a) Calculate the steady state solution for this initial-boundary value problem.
   (b) Solve this initial boundary value problem.

2. Consider the following initial-boundary value problem for $x \in [0,1]$:

$$
\begin{aligned}
&u_t = u_{xx} - u,\\
&u(0,t)\\
&u(1,t) = 0,\\
&u(x,0) = \sin(3\pi x).
\end{aligned}
$$

   (a) Using separation of variables solve this initial-boundary value problem.
   (b) Using your solution, calculate $\lim_{t\to\infty} u(x,t)$.
   (c) Are there any steady state solutions to this equation? If so, what are they?

3. Consider the following initial-boundary value problem for $x \in [0,\pi]$:

$$
\begin{aligned}
&u_{tt} = u_{xx},\\
&u_x(0,t) = 0\\
&u_x(\pi,t) = 0,\\
&u(x,0) = \cos^2(x),\\
&u_t(x,0) = \cos(3x).
\end{aligned}
$$

   (a) Solve this initial-boundary value problem. **Hint:** It might be useful to use trig identities to reduce $\cos^2(x)$.
   (b) Sketch the solution for $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.
   (c) Describe qualitatively the behavior of the solution.

4. Solve the following initial-boundary value problem for $x \in [0, \pi]$:

$$\begin{aligned} u_{tt} + u_t &= u_{xx}, \\ u(0,t) &= 0 \\ u(\pi,t) &= 0, \\ u(x,0) &= \sin^2(x), \\ u_t(x,0) &= 0. \end{aligned}$$

5. Solve the following boundary value problem on the domain $\Omega = [0,1] \times [0,1]$

$$\begin{aligned} \Delta u &= 0 \\ u(0,y) &= \sin(\pi y) \\ u(1,y) &= \sin(2\pi y) \\ u(x,0) &= \sin(3\pi x) \\ u(x,1) &= \sin(4\pi x). \end{aligned}$$

Homework #9

#1

Consider the initial-boundary value problem:

$$U_t = U_{xx}$$
$$U(0,t) = 1$$
$$U(2\pi,t) = 2$$
$$U(x,0) = 1$$

(a) Calculate the steady state solution for this initial-boundary value problem.

(b) Solve this initial-boundary value problem.

Solution:

(a) $U^*_{xx} = 0$

$$\Rightarrow U^*(x) = \frac{1}{2\pi} x + 1$$

(b) Letting $V = U - U^*$ it follows that

$$V_t = V_{xx}$$
$$V(0,t) = 0$$
$$V(2\pi,t) = 0$$
$$V(x,0) = -\frac{1}{2\pi} x$$

Seperating variables will yield

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t/4} \sin(nx/2)$$

$$\Rightarrow -\frac{1}{2\pi} x = \sum_{n=1}^{\infty} b_n \sin(nx/2)$$

$$\Rightarrow b_n = -\frac{1}{2\pi^2} \int_0^{2\pi} x \sin(nx/2)\, dx$$
$$= -\frac{1}{2\pi^2}\left(-2/n\, x \cos(nx/2)\Big|_0^{2\pi} + 2/n \int_0^{2\pi} \cancel{\cos(nx/2)}^{\,0}\, dx\right)$$
$$= \frac{2}{\pi n}(-1)^n$$

Therefore

$$U(x,t) = 1 + \frac{1}{2\pi} x + \sum_{n=1}^{\infty} \frac{2}{\pi n}(-1)^n e^{-n^2 t/4} \sin(nx/2).$$ ∎

#2

Consider the initial value problem for $x \in [0,1]$:

$$u_t = u_{xx} - u$$
$$u(0,t) = 0$$
$$u(1,t) = 0$$
$$u(x,0) = \sin(3\pi x).$$

(a) Using seperation of variables solve this intial-boundary value problem.

(b) Using your solution, calculate $\lim_{t \to \infty} u(x,t)$.

(c) Are there any steady state solutions? If so what are they.

Solution:

(a) Letting $u = X(x)T(t)$ we have

$$XT' = X''T - XT$$
$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} - 1$$
$$\Rightarrow \frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda$$

Therefore, from boundary conditions $\lambda = n^2\pi^2$ and

$$X_n = \sin(n\pi x)$$
$$T_n = e^{-(n^2\pi^2+1)t}$$

Consequently,

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2+1)t} \sin(n\pi x)$$

Initial conditions yield

$$u(x,t) = e^{-(9\pi^2+1)t} \sin(3\pi x)$$

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#3

Consider the following intial-boundary value problem for $x \in [0, \pi]$:

$$U_{tt} = U_{xx}$$
$$U_x(0,t) = 0$$
$$U_x(\pi,t) = 0$$
$$U(x,0) = \cos^2(x) = 1/2 + \cos(2x)/2$$
$$U_t(x,0) = \cos(3x).$$

(a) Solve this intial-boundary value problem

(b) Sketch the solution for $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$.

(c) Describe qualitatively the behavior of the solution.

<u>Solution:</u>

(a) Seperation of variables yields

$$u(x,t) = a_0 + a_1 t + \sum_{n=1}^{\infty} b_n \cos(nt)\cos(nx) + \sum_{n=1}^{\infty} c_n \sin(nt)\cos(nx)$$

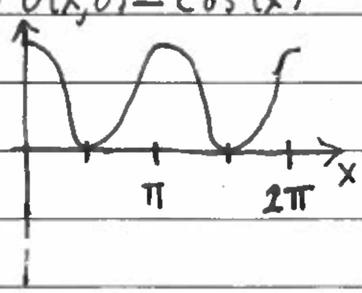
Initial conditions imply

$$a_0 = 1/2, \quad b_2 = 1/2, \quad c_3 = 1/3$$

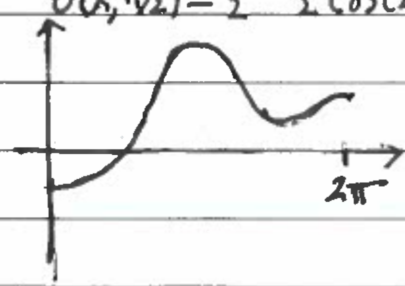
and all other terms are 0. Therefore,

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2t)\cos(2x) + \frac{1}{3}\sin(3t)\cos(3x).$$

(b) $u(x,0) = \cos^2(x)$



$u(x,\pi/2) = \frac{1}{2} - \frac{1}{2}\cos(2x) - \frac{1}{3}\cos(3x)$



and then the pattern repeats

(c) The solution "waves" with period $T = \pi$.

#4

Solve the following initial-boundary value problem for $x \in [0,\pi]$:

$$u_{tt} + u_t = u_{xx}$$
$$u(0,t) = 0$$
$$u(\pi,t) = 0$$
$$u(x,0) = \sin^2(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$
$$u_t(x,0) = 0$$

Solution:

Letting $u = X(x)T(t)$ we have

$$XT'' + XT' = X''T$$
$$\Rightarrow \frac{T'' + T'}{T} = \frac{X''}{X} = -\lambda$$

Boundary conditions imply that $\lambda = n^2$ and

$$X = \sin(nx).$$

Solving for T we have that

$$T'' + T' = -n^2 T$$
$$\Rightarrow T'' + T' + n^2 T = 0$$
$$\Rightarrow T = a e^{-\frac{1}{2}t}\cos\left(\tfrac{1}{2}\sqrt{n^2-1}\, t\right) + b e^{-\frac{1}{2}t}\sin\left(\tfrac{1}{2}\sqrt{n^2-1}\, t\right)$$

By linear superposition we have that

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n e^{-\frac{1}{2}t}\cos\left(\tfrac{1}{2}\sqrt{n^2-1}\, t\right) + b_n e^{-\frac{1}{2}t}\sin\left(\tfrac{1}{2}\sqrt{n^2-1}\, t\right)\right)\sin(nx).$$

Now,

$$u(x,0) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right)\sin(nx)\,dx$$
$$= \frac{1}{\pi}\left(\int_0^{\pi}\sin(nx)\,dx - \int_0^{\pi}\cos(2x)\sin(nx)\,dx\right)$$
$$= \frac{1}{\pi}\left(-\frac{1}{n}\left((-1)^n - 1\right) - \frac{1}{2}\left(\int_0^{\pi}\sin((2+n)x) - \sin((2-n)x)\,dx\right)\right)$$
$$= \begin{cases} \dfrac{1\left(1-(-1)^n\right)}{\pi n} + \dfrac{(-1)^n - 1}{2(2+n)\pi} - \dfrac{(-1)^n - 1}{2(2-n)\pi}, & n \neq 2 \\ 0, & n = 2 \end{cases}$$

Therefore,

$$a_n = \begin{cases} \dfrac{1-(-1)^n}{\pi}\left(1 - \dfrac{1}{2(2+n)} + \dfrac{1}{2(2-n)}\right), & n \neq 2 \\ 0, & n = 2 \end{cases}$$

$$= \begin{cases} \dfrac{1-(-1)^n}{\pi}\left(\dfrac{4+2n-\cancel{4}+2n+\cancel{4}+2n}{4(4-n^2)}\right), & n \neq 2 \\ 0, & n = 2 \end{cases}$$

$$= \begin{cases} \dfrac{1-(-1)^n}{\pi}\,\dfrac{(2+3n)}{2(2-n^2)}, & n \neq 2 \\ 0 & n = 2 \end{cases}$$

We also have that

$$U_t = -\tfrac{1}{2}U(x,t) + \left(\sum_{n=1}^{\infty} a_n/2\sqrt{n^2-1}\, e^{-\frac{1}{2}t} \sin\left(\tfrac{1}{2}\sqrt{n^2-1}\,t\right) + \sum_{n=1}^{\infty} -b_n/2\sqrt{n^2-1}\, e^{-\frac{1}{2}t} \cos\left(\tfrac{1}{2}\sqrt{n^2-1}\,t\right)\right)\sin(nx)$$

Therefore,

$$U_t(x,0) = 0 \Rightarrow 0 = -\tfrac{1}{2}\sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=1}^{\infty} -b_n/2\sqrt{n^2-1}\,\sin(nx)$$

$$\Rightarrow b_n = \frac{-a_n}{\sqrt{n^2-1}}, \quad n \neq 1. \quad \blacksquare$$

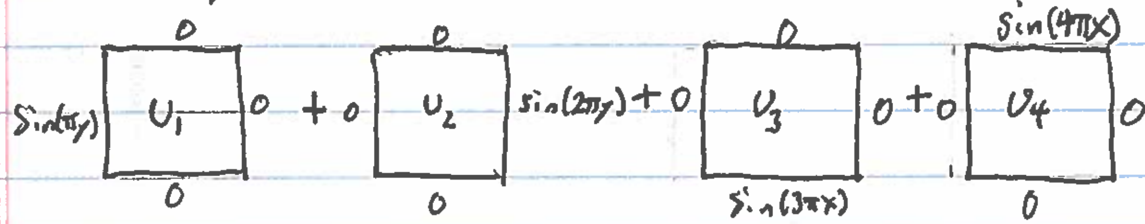
#5

Solve the following boundary value problem on the domain $\Omega = [0,1] \times [0,1]$

$$\Delta u = 0$$
$$u(0,y) = \sin(\pi y)$$
$$u(1,y) = \sin(2\pi y)$$
$$u(x,0) = \sin(3\pi x)$$
$$u(x,1) = \sin(4\pi x)$$

Solution:

Splitting into the four boundary value problems:



From the homogeneous boundary conditions:

$$u_1(x,y) = \sum_{n=1} a_n \sin(n\pi y)\sinh(n\pi(x-1))$$
$$u_2(x,y) = \sum_{n=1} b_n \sin(n\pi y)\sinh(n\pi x)$$
$$u_3(x,y) = \sum_{n=1} c_n \sinh(n\pi(y-1))\sin(n\pi x)$$
$$u_4(x,y) = \sum_{n=1} d_n \sinh(n\pi y)\sin(n\pi x)$$

From the inhomogeneous boundary condits we have that

$$a_1 = -\frac{1}{\sinh(\pi)},\quad b_2 = \frac{1}{\sinh(2\pi)},\quad c_3 = -\frac{1}{\sinh(3\pi)},\quad c_4 = \frac{1}{\sinh(4\pi)}$$

$$\Rightarrow u(x,y) = -\frac{\sinh(\pi(x-1))}{\sinh(\pi)}\sin(\pi y) + \frac{\sinh(2\pi x)}{\sinh(2\pi)}\sin(2\pi y)$$
$$-\frac{\sin(3\pi(y-1))}{\sinh(3\pi)}\sin(3\pi x) + \frac{\sinh(4\pi y)}{\sinh(4\pi)}\sin(4\pi x).$$

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