MTH 352/652
Homework #10

Due Date: April 25, 2025

1. Solve the following boundary value problem on a quarter wedge of radius R:

$$\Delta u = 0,\ \Omega = \{(r,\theta) : r < R,\ 0 \leq \theta \leq \pi/2\},$$
$$u(R,\theta) = \sin(2\theta),$$
$$u(r,0) = u(r,\pi/2) = 0.$$

2. Solve the following boundary value problem on an annulus:

$$\Delta u = 0,\ \Omega = \{(r,\theta) : 1 < r < 2,\ 0 \leq \theta \leq 2\pi\},$$
$$u(1,\theta) = 0,$$
$$u(2,\theta) = \sin^2(\theta).$$

3. Consider the wave equation on the *real line* $x \in \mathbb{R}$, $t > 0$ with a source term:

$$u_{tt} = c^2 u_{xx} + \sin(x),$$
$$u(x,0) = 0,$$
$$u_t(x,0) = 0.$$

   (a) Find all steady state solutions to this problem.

   (b) Solve this initial value problem.

4. Using Duhamel's principle, find a formula for the solution to the following initial value problem on the real line $x \in \mathbb{R}$, $t > 0$:

$$u_t + cu_x = f(x,t),$$
$$u(x,0) = 0.$$

5. Solve the following initial value problem on the real line $x \in \mathbb{R}$, $t > 0$:

$$u_t + 2u_x = xe^{-t},$$
$$u(x,0) = 0.$$

6. Consider the following initial boundary value problem on the domain $x \in [0,1], t > -0$:

$$
\begin{aligned}
u_t &= k u_{xx}, \\
u(0,t) &= \sin(t), \\
u(1,t) &= 1, \\
u(x,0) &= \sin(\pi x) + x.
\end{aligned}
$$

   (a) Give a physical interpretation of the boundary conditions for this problem.

   (b) Transform this problem into a problem with homogeneous boundary conditions.

   (c) Solve this initial-boundary value problem.

7. Solve the following initial boundary value problem for $x \in [0,1]$, $t > 0$:

$$
\begin{aligned}
u_t &= u_{xx} + \sin(3\pi x), \\
u(0,t) &= 0, \\
u(1,t) &= 0, \\
u(x,0) &= \sin(\pi x).
\end{aligned}
$$

8. Consider the following wave equation with a constant gravitational force g modeling the dynamics of rope of length L:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} - g, \\
u(0,t) &= 0, \\
u(L,t) &= \sin(t), \\
u(x,0) &= x(L-x), \\
u_t(x,0) &= 0.
\end{aligned}
$$