Lecture #15: Wave Equation

Example:

Solve:

$$u_{tt} = u_{xx}$$ ← wave equation with speed 1.

$$u(0,t) = 0$$
$$u(L,t) = 0$$

wave is fixed at 0 at $x = 0, L$

$$u(x,0) = f(x)$$ ← initial position

$$u_t(x,0) = 0$$ ← initial displacement

Let $u(x,t) = X(x)T(t)$

$$\Rightarrow T''X = X''T$$

$$\Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$
$$T = C\cos(\sqrt{\lambda}t) + D\sin(\sqrt{\lambda}t)$$

Boundary conditions imply

$$X = B\sin(\sqrt{\lambda}x) \text{ and } \lambda = n^2\pi^2/L.$$

Therefore, a generic solution is given by

$$u_n(x,t) = \left(c_n\cos(n\pi/L\, t) + d_n\sin(n\pi/L\, t)\right)\sin(n\pi/L\, x).$$

Linear superposition:

$$u(x,t) = \sum_{n\geq 1}^{\infty}\left(c_n\cos(n\pi/L\, t) + d_n\sin(n\pi/L\, t)\right)\sin(n\pi/L\, x).$$

Initial Conditions imply

$$u_t(x,0) = 0 \Rightarrow d_n = 0$$

$$u(x,0) = f(x) = \sum_{n\geq 1}^{\infty} c_n \sin(n\pi/L\, x)$$

$$\Rightarrow \boxed{c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx}$$

$$\Rightarrow \boxed{u(x,t)=\sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

Each principle wave

$$u_n(x,t)=\cos\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

has a spatial wavelength

$$\Lambda_n=\frac{2L}{n}$$

and a temporal period

$$T_n=\frac{2L}{n}$$

Therefore, the temporal period of u is given by

$$T=\max_n \Lambda_n=2L.$$

The fact that $\Lambda_n=T_n$ is called a dispersion relationship.

Theorem- If we let $E(t)=\int_0^L\left(\frac{1}{2}u_x^2+\frac{1}{2}u_t^2\right)dx$ then for solutions to * $E(t)$ is conserved $\Rightarrow$ Solutions are unique.

proof:

$$\frac{dE}{dt}=\int_0^L(u_x u_{xt}+u_t u_{tt})dx$$

$$=\cancelto{0}{+u_t u_x\Big|_0^L}-\int_0^L u_{xx}u_t\,dx+\int_0^L u_t u_{tt}\,dx$$

$$=\int_0^L u_t\,\cancelto{0}{(u_{tt}-u_{xx})}\,dx$$

$$=0$$

■

Example:

$u_{tt} = u_{xx}$ ← Wave equation with speed 1

$u_x(0,t)=0$, ** $u_x(L,t)=0$ ← wave/string held flat at $x=0, L$

$u(x,0)=0$ ← zero initial displacemt

$u_t(x,0)=f(x)$ ← initial velocity

Let $u(x,t) = X(x)T(t)$

$$\Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow \left.\begin{aligned} X &= A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \\ T &= C\cos(\sqrt{\lambda}t) + D\sin(\sqrt{\lambda}t) \end{aligned}\right\} \lambda \neq 0$$

$$\left.\begin{aligned} X &= ax+b \\ T &= ct+d \end{aligned}\right\} \lambda = 0$$

From boundary conditions

$X = a_n \cos(n\pi/L\, x)$, $n \neq 0$

$X = a_0$, $n=0$ (i.e, $\lambda = 0$)

Linear superposition:

$$u(x,t) = ct + d + \sum_{n=1}^{\infty} \left(c_n \cos(n\pi/L\, t) + d_n \sin(n\pi/L\, t)\right)\cos(n\pi x/L)$$

Initial conditions

$$u(x,0) = 0 = d + \sum_{n=1}^{\infty} c_n \cos(n\pi x/L)$$

$\Rightarrow d=0$ and $c_n = 0$

Initial velocity:

$$u_t(x,0) = f(x) = c + \sum_{n=1}^{\infty} d_n\, n\pi/L \cos(n\pi x/L).$$

Therefore,

$$\boxed{\begin{aligned} c &= \tfrac{1}{L}\int_0^L f(x)\,dx \\ d_n &= \tfrac{2}{n\pi}\int_0^L f(x)\cos\left(\tfrac{n\pi x}{L}\right)dx \end{aligned}}$$

$$\Rightarrow \boxed{u(x,t) = ct + \sum_{n=1}^{\infty} d_n \cos\left(\tfrac{n\pi t}{L}\right)\cos\left(\tfrac{n\pi x}{L}\right)}$$

Example:

$$u_{tt} = u_{xx}$$
$$u(0,t) = 0$$
$$u(\pi,t) = 0$$
$$u(x,0) = \sin^3(x)$$
$$u_t(x,0) = \sin(5x)$$

From boundary conditions we know

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))\sin(nx)$$

From intial conditions

$$u(x,0) = \tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\Rightarrow a_1 = \tfrac{3}{4},\ a_3 = -\tfrac{1}{4} \text{ all other } 0.$$

$$u_t(x,0) = \sin(5x) = \sum_{n=1}^{\infty} n b_n \sin(nx)$$

$$\Rightarrow b_5 = \tfrac{1}{5} \text{ all other } 0$$

$$\Rightarrow \boxed{u(x,t) = \frac{3}{4}\cos(t)\sin(x) - \frac{1}{4}\cos(3t)\sin(3x) + \frac{1}{5}\sin(5t)\sin(5x).}$$