Lecture #16: Laplaces Equation Part I

The operator Δ is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad u: \mathbb{R}^2 \to \mathbb{R}$$

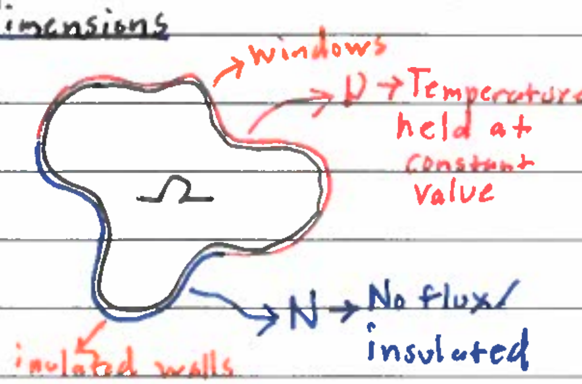
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad u: \mathbb{R}^3 \to \mathbb{R}$$

The heat equation in higher dimensions

$$u_t = \Delta u, \quad \vec{x} \in \Omega$$

$$u = f(s), \quad \vec{x} \in \partial\Omega \cap D$$

$$\nabla \vec{u} \cdot \vec{n} = 0, \quad \vec{x} \in \partial\Omega \cap N$$



Steady state solutions satisfy Laplace's Equation

$$\Delta u = 0$$

$$u = f(s), \quad \vec{x} \in \partial\Omega \cap D$$

$$\nabla \vec{u} \cdot \vec{n} = 0, \quad \vec{x} \in \partial\Omega \cap N$$

Example:

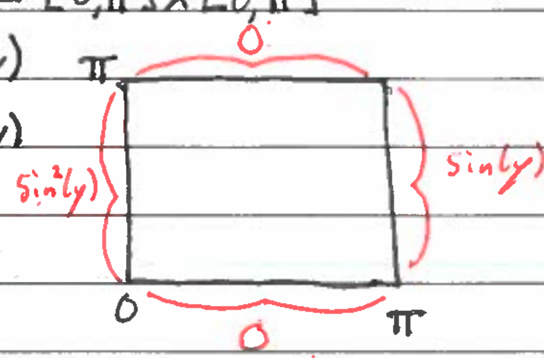
$$\Delta u = 0, \quad \Omega = [0,\pi] \times [0,\pi]$$

$$u(0,y) = \sin^3(y)$$

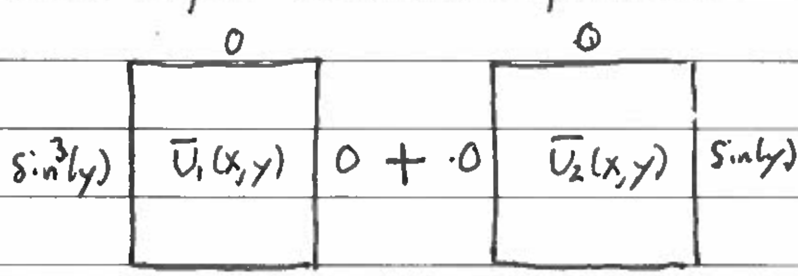
$$u(\pi,y) = \sin(y)$$

$$u(x,0) = 0$$

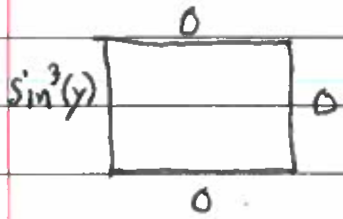
$$u(x,\pi) = 0$$



Idea: Split into two problems



Problem #1



Let $u = XY$

$\Rightarrow X''Y + XY'' = 0$

$\dfrac{Y''}{Y} = -\dfrac{X''}{X} = -\lambda$ → We choose $-\lambda$ to have periodic solutions in y.

Therefore,

$Y = A\cos(\sqrt{\lambda}y) + B\sin(\sqrt{\lambda}y)$

Boundary conditions imply $A = 0$, $\lambda = n^2$.

$\Rightarrow Y_n = B_n \sin(ny)$

We now have:

$X_n'' = n^2 X_n$

We choose as a solution:

$X_n = C\cosh(n(x-\pi)) + D\sinh(n(x-\pi))$

to simplify analysis of boundary conditions.

$X_n(\pi) = 0 \Rightarrow C = 0$

Therefore,

$u_n(x,y) = b_n \sinh(n(x-\pi))\sin(ny)$

$$\Rightarrow \tilde{u}(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n(x-\pi))\sin(ny)$$

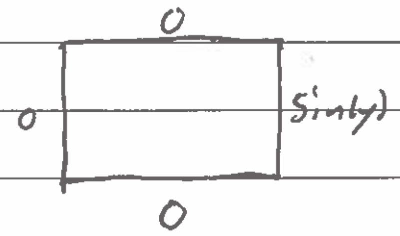
$$u_1(0,y) = \sin^3(y) = \frac{3}{4}\sin(3y) - \frac{1}{4}\sin(y) = \sum_{n=1}^{\infty} -b_n \sinh(n\pi)\, si(ny)$$

$\Rightarrow b_3 = -\dfrac{3}{4\sinh(3\pi)}$, $b_1 = \dfrac{1}{4\sinh(\pi)}$, all others zero.

Therefore,

$$u_1(x,y) = -\frac{3}{4\sinh(3\pi)}\sinh(3(x-\pi))\sin(3y) + \frac{1}{4\sinh(\pi)}\sinh((x-\pi))\sin(y)$$

Problem #2:



Let $U_2 = X \cdot Y$. Therefore,

$$X''Y + XY'' = 0$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

Boundary conditions imply

$$Y_n = b_n \sin(ny), \quad \lambda = n^2$$

$$\Rightarrow X_n'' = n^2 X_n$$

$$\Rightarrow X_n = c_n \cosh(nx) + d_n \sinh(nx)$$

Boundary conditions imply $c_n = 0$. Therefore,

$$U_n(x,y) = b_n \sinh(nx)\sin(ny)$$

Consequently,

$$\bar{U}_2(x,y) = \sum_{n=1}^{\infty} b_n \sinh(nx)\sin(ny)$$

$$\bar{U}_2(\pi, y) = \sin(y) \Rightarrow b_1 = 1/\sinh(\pi), \text{ all others are zero.}$$

Therefore,

$$\bar{U}_2(x,y) = \frac{1}{\sinh(\pi)} \sinh(x)\sin(y).$$

Finally,

$$U(x,y) = \bar{U}_1(x,y) + \bar{U}_2(x,y)$$

$$= -\frac{3\sinh(3(x-\pi))}{4\sinh(3\pi)}\sin(3y) + \frac{\sinh(x-\pi)}{4\sinh(\pi)}\sin(y) + \frac{\sinh(x)}{\sinh(\pi)}\sin(y).$$

Theorem (Maximum Principle) - If u solves Laplace's equation then u obtains its maximum on $\partial\Omega$.

proof (2-D):

If u obtains max(min) in Ω at (a,b) then $\nabla u|_{(a,b)}=0$.
The discrimenant is

$$u_{xx}u_{yy}-u_{xy}^2=\det\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}=\lambda_1\cdot\lambda_2=\text{product of eigenvalues}$$

$$u_{xx}+u_{yy}=\mathrm{tr}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}=u_{xx}+u_{yy}=\lambda_1+\lambda_2=0$$

$\Rightarrow \lambda_1=-\lambda_2$

$\Rightarrow \lambda_1\cdot\lambda_2<0$

$\Rightarrow$ discrimenant <0

$\Rightarrow$ saddle point $\Rightarrow$ cannot be a maximum or minimum. ∎

(More technical proof needed in higher dimensions, but this is the idea of the proof.)

Theorem - Solutions to Laplace's equation with Dirichlet boundary conditions are unique.

proof:

Suppose u_1, u_2 solve Laplace's equation. Therefore, $v=u_1-u_2$ solves Laplace's equation

$$v_{xx}+v_{yy}=0$$
$$v|_{\partial\Omega}=0$$

Maximum principle $\Rightarrow v=0 \Rightarrow u_1=u_2$. ∎