Lecture #17: Laplace's Equation Part II

Polar Coordinates:

$x = r\cos\theta$, $r^2 = x^2 + y^2$

$y = r\sin\theta$, $\theta = \tan^{-1}(y/x)$

Therefore,

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \cos\theta$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2}\,\frac{-y}{x^2} \Rightarrow \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2}\,\frac{1}{x} \Rightarrow \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Consequently,

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta}$$

$$+ \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} - 2\frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

We also have that

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$$

$$= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta}$$

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Therefore,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Example:

$$\Delta u = 0, \quad \Omega = \{x^2+y^2 \le 4\}$$

$$u|_{\partial\Omega} = \sin^2\theta$$

Converting to polar coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$u(2,\theta) = \sin^2\theta$$

$$u(r,0) = u(r,2\pi)$$

$$u_\theta(r,0) = u_\theta(r,2\pi)$$

Letting $u = R(r)\Theta(\theta)$ we have that

$$R''\Theta(\theta) + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$-\frac{r^2R'' + rR'}{R} = \frac{\Theta''}{\Theta} = -\lambda$$

Boundary conditions on θ imply

$$\Theta = a_n\cos(n\theta) + b_n\sin(n\theta), \quad \lambda = n^2$$

$$\Theta = a_0 \quad , \quad \lambda = 0.$$

The differential equation for R is therefore

$$r^2R'' + rR' = n^2R$$

$$\Rightarrow r^2R'' + rR' - n^2R = 0$$

Case 1 $(n=0)$:

$$r^2R'' + rR' = 0$$

Let $v = R'$

$$\Rightarrow r^2\frac{dv}{dr} + rv = 0$$

Make a guess $v = cr^p$

$$\Rightarrow r^2 p r^{p-1} + r r^p = 0$$

$$\Rightarrow p r^{p+1} + r^{p+1} = 0$$

$$\Rightarrow p = -1$$

$$\Rightarrow v = cr^{-1}$$

Therefore,

$$R' = cr^{-1}$$

$$\Rightarrow R = c\ln(r) + d$$

To enforce boundedness at $r=0$ we set $c=0$.

Case 2 ($n \neq 0$):

$$r^2R'' + rR' - n^2R = 0$$

Make a guess

$$R = r^p$$

$$\Rightarrow r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p = 0$$

$$\Rightarrow p(p-1) r^p + p r^p - n^2 r^p = 0$$

$$\Rightarrow p^2 - p + p - n^2 = 0$$

$$\Rightarrow p = \pm n.$$

Consequently,

$$R(r) = cr^n + dr^{-n}$$

To enforce boundedness at $r=0$ we set $d=0$.

By linear superposition:

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta).$$

Applying boundary conditions at $r=2$ we have

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} = a_0 + \sum_{n=1}^{\infty} a_n 2^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n 2^n \sin(n\theta).$$

Therefore,

$a_0 = 1/2$, $4a_2 = -1/2 \Rightarrow a_2 = -1/8$, all other coefficients are zero.

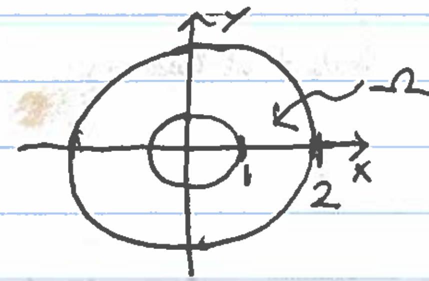
$$\Rightarrow u(r,\theta) = \frac{1}{2} - \frac{r^2}{8}\cos(2\theta).$$

Example:

$\Delta u = 0$, $\Omega = \{1 \le x^2 + y^2 \le 4\}$

$u(1, \theta) = \sin(3\theta)$

$u(2, \theta) = \cos(5\theta)$.



We have to split into two problems:

p1:

$\Delta u_1 = 0$

$u_1(1, \theta) = \sin(3\theta)$

$u_1(2, \theta) = 0$

$\downarrow$

To match boundary conditions:

$$u_1(r, \theta) = (a_1 r^3 + b_1 r^{-3}) \sin(3\theta)$$

Therefore,

$$a_1 + b_1 = 1$$

$$8a_1 + \frac{b_1}{8} = 0$$

$$\Rightarrow a_1 = -\frac{b_1}{64}$$

$$\Rightarrow b_1 - \frac{b_1}{64} = 1$$

$$\Rightarrow b_1 = \frac{64}{63}$$

$$\Rightarrow a_1 = \frac{-1}{63}$$

p2:

$\Delta u = 0$

$u(1, \theta) = 0$

$u(2, \theta) = \cos(5\theta)$

$\searrow$

To match boundary conditions:

$$u_2(r, \theta) = (a_2 r^5 + b_2 r^{-5}) \cos(5\theta)$$

Therefore,

$$a_2 + b_2 = 0$$

$$32a_2 + \frac{b_2}{32} = 1$$

$$\Rightarrow 32a_2 - \frac{a_2}{32} = 1$$

$$\Rightarrow (a_2 - 1)(2^5 - \tfrac{1}{2^5}) = 1$$

$$\Rightarrow a_2 = \frac{32}{1023}.$$

$$b_2 = -\frac{32}{1023}$$

Example:

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad \Omega = \{(r,\theta) : r \leq 1, 0 < \theta < \pi/4\}$

$u(1,\theta) = \sin(8\theta)$

$u(r,0) = 0$

$u(r,\pi/4) = 0$

Assuming $u = R\Theta$

$$\Rightarrow \frac{-r^2 R'' + r R'}{R} = \frac{\Theta''}{\Theta} = -\lambda$$

Boundary conditions on Θ imply

$$\Theta = b_n \sin(\sqrt{-\lambda}\,\theta)$$

$$\Theta(\pi/4) = b_n \sin(\sqrt{-\lambda}\,\pi/4) = 0$$

$$\Rightarrow \sqrt{-\lambda}\,\frac{\pi}{4} = n\pi$$

$$\Rightarrow \lambda = -16n^2$$

Solving for R we have

$$-r^2 R'' + r R'' + 16n^2 = 0$$

Solve as we did before,

$$R_n = a_n r^{4n}.$$

Consequently,

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{4n} \sin(4n\theta)$$

Boundary conditions imply

$$u(1,\theta) = \sin(8\theta) = \sum_{n=1}^{\infty} a_n \sin(4n\theta)$$

$\Rightarrow a_2 = 2$, all others are zero.

$\Rightarrow u(r,\theta) = r^8 \sin(8\theta).$