Lecture #18: Sources and Duhamel's Principle

Example:

$$u_t = u_{xx} + \sin^2(x)$$

$$u(0,t) = 0 \qquad \text{spatial source of heat}$$

$$u(\pi,t) = 0$$

$$u(x,0) = \sin(3x)$$

initial heat

This problem is inhomogeneous so seperation of variables will not work. Instead, find a particular solution - the steady state solution - denoted $u^*(x)$:

$$u^*_{xx} + \frac{1-\cos(2x)}{2} = 0$$

$$\Rightarrow u^*_{xx} = -\frac{1}{2} + \frac{\cos(2x)}{2}$$

$$\Rightarrow u^*(x) = -\frac{x^2}{4} - \frac{\cos(2x)}{8} + ax + b$$

Now,

$$u^*(0) = -\frac{1}{8} + b = 0$$

$$\Rightarrow b = 1/8$$

$$u^*(\pi) = -\frac{\pi^2}{4} - \frac{1}{8} + a\pi + \frac{1}{8}$$

$$a = \frac{\pi}{4}$$

Therefore,

$$u^*(x) = \frac{x^2}{4} - \frac{\cos(2x)}{8} + \frac{\pi}{4}x + \frac{1}{8}$$

If we let $v = u - u^*$ it follows that

$$v_t = v_{xx}$$
$$v(0,t) = 0$$
$$v(\pi,t) = 0$$
$$v(x,0) = \sin(3x) + \frac{x^2}{4} + \frac{\cos(2x)}{8} - \frac{\pi}{4}x - \frac{1}{8}$$

Therefore, $v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\sin(3x) + \frac{x^2}{4} + \frac{\cos(2x)}{8} - \frac{\pi}{4}x - \frac{1}{8}\right)\sin(nx)\,dx$$

Example (ODE)

How do we solve

$$y'(t) + ay(t) = F(t),\ t>0,\ y(0)=0$$

Multiply by e^{at}

$$\frac{d}{dt}\left(e^{at}y(t)\right) = e^{at}F(t)$$

$$\Rightarrow \int_0^t \frac{d}{d\tau}\left(e^{a\tau}y(\tau)\right) = \int_0^t e^{a\tau}F(\tau)\,d\tau$$

$$\Rightarrow e^{at}y(t) = \int_0^t e^{a\tau}F(\tau)\,d\tau$$

$$\Rightarrow y(t) = \int_0^t e^{-a(t-\tau)}F(\tau)\,d\tau$$

Example (Auxilliary Problem)

How do we solve

$$\frac{d}{dt} w(t;\tau) + a w(t;\tau) = 0, \quad t>0, \quad w(0;\tau) = F(\tau)$$

$$\Rightarrow \frac{d}{dt} w(t;\tau) = -a w(t;\tau)$$

$$\Rightarrow w(t;\tau) = F(\tau) e^{-at} = e^{-at} F(\tau).$$

Duhamel's Principle:

From the last two examples:

$$\boxed{y(t) = \int_0^t w(t-\tau, \tau)\, d\tau} \longrightarrow \text{Duhamel's principle.}$$

Example:

Solve

$$U_t + c U_x = f(x,t)$$
$$U(x,0) = g(x)$$

Problem #1:

$$\bar{U}_t + c\bar{U}_x = 0$$
$$\bar{U}(x,0) = g(x)$$

$$\Rightarrow \bar{U}(x,t) = g(x-ct).$$

Problem #2:

$$\tilde{U}_t + c\tilde{U}_x = f(x,t)$$
$$\tilde{U}(x,0) = 0$$

Auxilliary Problem:

$$W_t(x,t;\tau)+cW_x(x,t;\tau)$$
$$W(x,0;\tau)=f(x,\tau)$$
$$\Rightarrow W(x,t;\tau)=f(x-ct,\tau)$$

Therefore,

$$\tilde{U}(x,t)=\int_0^t W(t-\tau;\tau)\,d\tau$$

$$\Rightarrow \tilde{U}(x,t)=\int_0^t f(x-c(t-\tau),\tau)\,d\tau$$

The full solution is

$$\boxed{u(x,t)=\bar{U}(x,t)+\tilde{U}(x,t)=g(x-ct)+\int_0^t f(x-c(t-\tau),\tau)\,d\tau}$$

Why does this work?

$$U_t+cU_x=f(x,t)$$
$$u(x,0)=0$$

Apply Fourier transforms:

$$\hat{U}_t+ikc\hat{U}=\hat{f}(k,t)$$
$$\hat{U}(k,0)=0$$

We can Duhamel's principle to the ODE.

Example:

$$U_t = U_{xx} + \sin(x)\cos^2(t)$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = 0$$

Auxilliary Problem:

$$W_t = W_{xx}$$
$$W(0,t;\tau) = 0$$
$$W(\pi,t;\tau) = 0$$
$$W(x,0;\tau) = \sin(x)\cos^2(\tau)$$

$$\Rightarrow W(x,t;\tau) = \cos^2(\tau)e^{-t}\sin(x).$$

Therefore,

$$u(x,t) = \int_0^t W(x,t-\tau;\tau)\,d\tau$$
$$= \int_0^t \cos^2(\tau)e^{-(t-\tau)}\sin(x)\,d\tau$$

$$\boxed{u(x,t) = \frac{1}{10}\left(5 - 6e^{-t} + \cos(2t) + 2\sin(2t)\right)\sin(x)}$$

Example

$$u_{tt} = u_{xx} + \sin(2x)\sin(2t)$$
$$u(0,t)=0$$
$$u(\pi,t)=0$$
$$u(x,0)=0$$
$$u_t(x,0)=0$$

Auxilliary Problem:

$$W_{tt} = W_{xx}$$
$$w(0,t;\tau)=0$$
$$w(\pi,t;\tau)=0$$
$$w(x,0;\tau)=0$$
$$w_t(x,0;\tau)=\underline{\sin(2x)\sin(2\tau)}$$

← initial condition placed on initial velocity term.

From boundary conditions

$$w(x,t;\tau) = \sum_{n=1}^{\infty} b_n \sin(nt)\sin(nx)$$

$$\Rightarrow w(x,t;\tau) = \frac{\sin(2\tau)}{2}\sin(2t)\sin(2x)$$

Therefore,

$$u(x,t) = \frac{1}{2}\int_0^t \sin(2\tau)\sin(2(t-\tau))\sin(2x)\,d\tau$$

Why Does this Work:

Letting $v = u_t$ we have:

$$\begin{cases} v_t = u_{xx} + \sin(2x)\sin(2t) \\ v(x,0)=0 \end{cases} \qquad u_t = v \qquad u(x,0)=0$$

$$u(0,t)=0$$

We are applying Duhamel's principle to v. $\qquad u(\pi,t)=0$

Example:

$U_t = U_{xx}$

$u(0,t) = 2\sin(t)$ } 2 inhomogeneities

$u(1,t) = 1$

$u(x,0) = x^2$

Let $v(x,t) = u(x,t) - A(t) - B(t)x$

$\Rightarrow v_t = u_t - A'(t) - B'(t)x$

$v_{xx} = u_{xx}$

$\Rightarrow v_t = v_{xx} - A'(t) - B'(t)x$

$v(0,t) = 2\sin(t) - A(t)$

$v(1,t) = 1 - A(t) - B(t)$

$v(x,0) = x^2 - A(0) - B(0)x$

If we let $A(t) = 2\sin(t)$, $B(t) = 1 - 2\sin(t)$ we have that

$v_t = v_{xx} - 2\cos(t) + 2\cos(t)x$

$v(0,t) = 0$

$v(1,t) = 0$

$v(x,0) = x^2 - 0 - 0\cdot x = x^2$

We need to split into two problems

Problem #1

$\bar{V}_t = \bar{V}_{xx}$

$\bar{V}(0,t)=0$

$\bar{V}(1,t)=0$

$\bar{V}(x,0)=x^2$

$$\Rightarrow \bar{V}(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$b_n = 2\int_0^1 x^2 \sin(n\pi x)$$

Problem #2

$\hat{V}_t = \hat{V}_{xx} - 2\cos(t) + 2\cos(t)x$

$\hat{V}(0,t)=0$

$\hat{V}(1,t)=0$

$\hat{V}(x,0)=0$

Auxilary Problem:

$W_t = W_{xx}$

$W(0,t;\tau)=0$

$W(1,t;\tau)=0$

$W(x,0;\tau) = -2\cos(\tau) + 2\cos(\tau)x$

$$\Rightarrow W(x,t;\tau) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$C_n = 2\int_0^1 (-2\cos(\tau) + 2\cos(\tau)x)\sin(n\pi x)\,dx$$

$$\Rightarrow \tilde{U}(x,t) = \int_0^t W(x, t-\tau; \tau)\,d\tau$$

$$\Rightarrow U(x,t) = \bar{U}(x,t) + \tilde{U}(x,t)$$