

MTH 352

Quiz #8

1. Consider the following initial-boundary value problem for $x \in [0, 1]$:

$$\begin{aligned} u_{tt} &= u_{xx} - u, \\ u_x(0, t) &= 0, \\ u_x(1, t) &= 0. \end{aligned}$$

Note, I did not provide any initial conditions for this problem as they are not relevant to this problem. By assuming a solution of the form $u(x, t) = X(x)T(t)$, find all separable solutions to this boundary value problem.

Let $u = X(x)T(t)$. Therefore,

$$\begin{aligned} T''X &= TX'' - TX \\ \Rightarrow \frac{T''}{T} + 1 &= \frac{X''}{X} = -\lambda \end{aligned}$$

Boundary conditions imply $\sqrt{\lambda} = n\pi, n \in \{0, 1, \dots\}$

Case 1 ($\lambda = 0$):

$$X = a_0$$

$$\frac{T''}{T} = -1$$

$$T = c_0 \cos(t) + d_0 \sin(t)$$

Case 2 ($\lambda^2 \geq n\pi$):

$$X = a_n \cos(n\pi x)$$

$$T'' = (\lambda - 1)T$$

$$T = c_n \cos(\sqrt{n^2\pi^2 + 1}t) + d_n \sin(\sqrt{n^2\pi^2 + 1}t)$$

Seperable solutions:

$$\{c_0 \cos(t) + d_0 \sin(t), (c_n \cos(\sqrt{n^2\pi^2 + 1}t) + d_n \sin(\sqrt{n^2\pi^2 + 1}t)) \cos(n\pi x)\}$$