Homework #4, Due Date: 03/20/26
MTH 752: Spring 2026
Instructor: John Gemmer

Problem 1. Consider the system

$$\dot{x} = (1 - x^2)(x - y),$$
$$\dot{y} = x + y(1 - x^2).$$

Draw a phase portrait for this system and use your phase portrait to determine the possible ω-limit sets of this system for all possible initial conditions. **Hint:** Recall from Colton's presentation that the ω-limit sets for a point $\mathbf{x}_0$ for a system of differential equations

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $F : \mathbb{R}^n \mapsto \mathbb{R}^n$, are defined as follows. If $\varphi(t, \mathbf{x}_0)$ denote the solution to this equation with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, then the ω-limit set of $\mathbf{x}_0$, denoted $\omega(\mathbf{x}_0)$, is

$$\omega(\mathbf{x}_0) = \left\{ \mathbf{y} \in \mathbb{R}^n : \text{there exists a subsequence of time } t_k \text{ such that } \lim_{k \to \infty} \varphi(t_k, \mathbf{x}_0) = \mathbf{y} \right\}.$$

Problem 2. Assuming $\varepsilon \ll 1$, find the first two nonzero terms in the asymptotic expansions for all possible solutions of the following equations:

(a) $\varepsilon x^3 - (x-1)(x-2) = 0,$

(b) $\varepsilon x^3 - (x-1)^2 = 0,$

(c) $\varepsilon x^4 - x^2 - x + 2 = 0.$

Problem 3. Assuming $\varepsilon \ll 1$, let $x(\varepsilon)$ be the solution near 0 of the equation

$$\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon.$$

Find the first two nonzero terms of an the asymptotic expansions for $x(\varepsilon)$.

Problem 4. A function $f(\varepsilon)$ is called transcendentally small if for all $p \in \mathbb{N}$, $f(\varepsilon) = o(\varepsilon^p)$, i.e., its decay rate to zero transcends the decay rate of all polynomials in ε. Prove that $f(\varepsilon) = e^{-1/\varepsilon}$ is transcendentally small. **Note:** This is a useful fact to keep in mind in this course.

Problem 5. In this problem it is assumed that $\varepsilon \ll 1$.

(a) Show that if $f = O(\varepsilon^\alpha)$, then $f = o(\varepsilon^\beta)$ for any $\beta < \alpha$.

(b) Show that if $f = O(g)$, then $f^\alpha = O(g^\alpha)$ for any positive α. Give an example to show that this result is not necessarily true if α is negative.

(c) Give an example to show that $f = O(g)$ does not necessarily imply that $e^f = O(e^g)$.

Problem 6. This problem establishes some of the basic properties of the order symbols, which are used extensively in this course. Again assume $\varepsilon \to 0$.

(a) If $f = o(g)$ and $g = O(h)$, or if $f = O(g)$ and $g = o(h)$, show that $f = o(h)$. Note that this result is often written as

$$o(O(h)) = O(o(h)) = o(h).$$

(b) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f + g = O(|\phi_1| + |\phi_2|)$. Note that this result is often written as

$$O(f) + O(g) = O(|f| + |g|).$$

(c) Assuming $f = O(\phi_1)$ and $g = O(\phi_2)$, show that $f \cdot g = O(\phi_1 \phi_2)$. This result is often written as

$$O(f)O(g) = O(fg).$$

(d) Show that $O(O(f)) = O(f)$.

(e) Show that $O(f)o(g) = o(f)o(g) = o(fg)$

Problem 7. In the following problems, sketch a generic phase portrait of the system for $\varepsilon \ll 1$ and then use the method of multiple scales to determine a first order uniform asymptotic expansions for the solution given arbitrary initial conditions:

(a) Drag model: $\ddot{x} + x = -\varepsilon|\dot{x}|\dot{x}$,

(b) Van der Pol oscillator: $\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$,

(c) Modified Van der Pol oscillator: $\ddot{x} + x = \varepsilon(1 - x^4)\dot{x}$,

(d) Van der Pol oscillator with cubic nonlinearity: $\ddot{x} + x = \varepsilon(1 - x^2)\dot{x} - \varepsilon x^3$.