Homework #6, Due Date: 04/29/26
MTH 752: Spring 2026
Instructor: John Gemmer

Problem 1. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[\gamma] = \int_0^1 \left(\dot{\gamma}(t)\sin\left(\pi\gamma(t)\right) - (t+\gamma(t))^2\right) dt,$$

where $\mathcal{A} = \{\gamma \in C^2([0,1];\mathbb{R}) : \gamma(0) = 0 \text{ and } \gamma(1) = -1\}$.

1. Prove that for all $\gamma \in \mathcal{A}$, $I[\gamma] \leq \frac{2}{\pi}$.
2. Find a $\gamma^* \in \mathcal{A}$ such that $I[\gamma^*] = \frac{2}{\pi}$ thus proving that this functional has a maximizer.

Problem 2. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[\gamma] = \int_{-1}^1 t^2\dot{\gamma}^2 dt,$$

where $\mathcal{A} = \{\gamma \in C^1([-1,1];\mathbb{R}) : \gamma(-1) = -1 \text{ and } \gamma(1) = 1\}$.

(a) Show that for all $\gamma \in \mathcal{A}$, $I[\gamma] > 0$.

(b) By constructing a sequence of functions $\gamma_n \in \mathcal{A}$ such that $\lim_{n\to\infty} I[\gamma_n] = 0$, prove that this functional has no minimizer in $\mathcal{A}$.

Problem 3. Let $\eta \in C^\infty([0,1];\mathbb{R})$. Find the Gateaux derivative of the following functionals in the direction of η, i.e., compute $\delta_\eta I[\gamma]$:

1. $I[\gamma] = \int_0^1 \gamma(t)\dot{\gamma}(t)dt$
2. $I[\gamma] = e^{\gamma(1)}$
3. $I[\gamma] = \int_0^1\int_0^1 F(s,t)\gamma(s)\gamma(t)dsdt$, where $F : \mathbb{R}^2 \mapsto \mathbb{R}$ is a given smooth function.
4. $I[\gamma] = \int_0^1 \dot{\gamma}(t)^2 dt + G(\gamma(1))$, where G is a given smooth function.

Problem 4. The second variation of a functional $I : \mathcal{A} \mapsto \mathbb{R}$ at $\gamma^* \in \mathcal{A}$ in the direction of $\eta \in C^\infty([0,1];\mathbb{R})$ is defined by

$$\delta_\eta^2 I[\gamma^*] = \lim_{\varepsilon\to 0}\frac{d^2}{d\varepsilon^2} I[\gamma^* + \varepsilon\eta].$$

Find the second variation of the functional

$$I[\gamma] = \int_0^1 \left(t\dot{\gamma}(t)^2 + \gamma(t)\sin\left(\dot{\gamma}(t)\right)\right) dt,$$

where $\mathcal{A} = C^2([0,1];\mathbb{R})$.

Problem 5. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[\gamma] = \int_0^1 L(t, \gamma, \dot{\gamma}(t))dt,$$

where $\mathcal{A} = \{\gamma \in C^2([0,1];\mathbb{R})\}$. Show that the Euler-Lagrange equations can be equivalently written in the form

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}}\right) = 0.$$

Problem 6. Suppose $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a vector field and the standard coordinates in $\mathbb{R}^n$ are denoted $x_1, \ldots, x_n$. Assuming Einstein summation notation, simplify the following:

(a) δ_{mm}

(b) $\delta_{ij}\delta_{ki}$

(c) $\delta_{ij}\mathbf{F}_{k,kj}$

(d) $\delta_{ij}\delta_{ij}$

(e) $\delta_{mm}\delta_{ij}x_j$

(f) $\delta_{km}\mathbf{F}_{i,jk}\mathbf{F}_{i,jm}$

(g) $\dfrac{\partial x_m}{\partial x_k}\dfrac{\partial x_m}{\partial x_k}$

Problem 7. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are vectors and ε_{ijk} is the permutation symbol, i.e. the Levi-Civita symbol, introduced in class.

(a) Explain why for all $i, j, k \in \{1, 2, 3\}$ it follows that $\varepsilon_{ijk} = -\varepsilon_{jik}$ and $\varepsilon_{ijk} = -\varepsilon_{ikj}$.

(b) Explain why the following identity is true

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

(c) Use summation notation to prove that

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

(d) Use summation notation to prove that

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \times \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle = -\langle \mathbf{u}, \mathbf{w} \times \mathbf{v} \rangle = -\langle \mathbf{v}, \mathbf{u} \times \mathbf{w} \rangle = -\langle \mathbf{w}, \mathbf{v} \times \mathbf{u} \rangle.$$

(e) Use summation notation to prove that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}.$$

Problem 8. Suppose $\mathbf{F} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a vector field, $f : \mathbb{R}^3 \mapsto \mathbb{R}$ is a scalar field, $A, B, C \in \mathbb{R}^{3\times 3}$ are matrices and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are vectors. Express the following mathematical operations in Einstein summation notation.

(a) $C = AB$

(b) $A^T B - AB^T$

(c) $\mathbf{u} = A^T \mathbf{v}$

(d) $\nabla \cdot \mathbf{F}$

(e) $\nabla^2 f$

(f) $\nabla^2 \mathbf{F}$

(g) ∇f

(h) $\nabla \times \mathbf{F}$

(i) $\nabla \times \nabla \times \mathbf{F}$

(j) $\nabla \cdot \nabla \times \mathbf{F}$

(k) $\nabla \times \nabla f$

(l) $\nabla(\nabla \cdot \mathbf{F})$

(m) $\nabla \cdot \nabla f$

Problem 9. Suppose $\mathbf{F}, \mathbf{G} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ are vector fields and $f, g : \mathbb{R}^3 \mapsto \mathbb{R}$ are scalar fields. You can assume that $\mathbf{F}, \mathbf{G}, f, g$ are regular enough that the order of all of its partial derivatives can be switched, i.e., Clairaut's theorem applies. Use summation notation to prove the following

(a) $\nabla(fg) = (\nabla f)g + f(\nabla g)$

(b) $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot F) + \langle \mathbf{F}, \nabla f \rangle$

(c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \langle \mathbf{G}, \nabla \times \mathbf{F} \rangle - \langle \mathbf{F}, \nabla \times \mathbf{G} \rangle$

(d) $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) - F \times (\nabla f)$

(e) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(f) $\nabla \times (\nabla f) = 0$

(g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \Delta\mathbf{F}$, where the Laplacian operation applied to F is defined in components by

$$\Delta F = F_{i,jj}.$$

Problem 10. Consider a particle of mass m constrained to lie on a vertically aligned ellipse:

$$c = \{(x(s), 0, z(s)) = (a\sin(s), 0, b\cos(s)) : s \in [0, 2\pi]\}$$

in an external constant gravitational field so that the potential energy is $V(x, y, z) = mgz$.

(a) Assuming the standard kinetic energy, find the Lagrangian for this system in terms of the generalized coordinate s.

(b) Find the Euler-Lagrange equations for the coordinate s

(c) Use the Legendre transformation to transform the Lagrangian into Hamilton's form and obtain Hamilton's equations. Check that these reduce to those the planar pendulum when $a = b$.

Problem 11. Consider a particle of mass m moving without friction that is constrained to lie on a two-dimensional surface specified by $z = F(x, y)$.

(a) Obtain the Lagrangian for this system assuming that the kinetic energy is the standard form and the gravitational potential energy is $V = mgz$.

(b) Derive the Euler-Lagrange equations for the coordinates x and y. Solve for $\ddot{x}$ and $\ddot{y}$ in terms of $\dot{x}, \dot{y}, x, y$.

(c) Suppose that $Z(x, y) = f(x - y)$. Show that the momentum $p = p_x + p_y$ is conserved. Here the momenta are defined by $p_x = \frac{\partial L}{\partial \dot{x}}, p_y = \frac{\partial L}{\partial \dot{y}}$. **Hint:** It is much easier to use the fact that the Lagrangian has the form $L(x - y, \dot{x}, \dot{y})$ then to use the differential equations you found in part b.