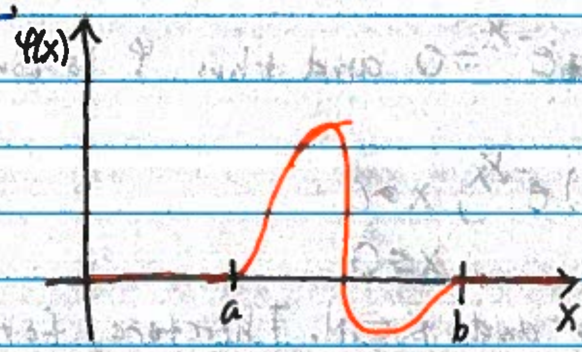
Lecture #10: Functions with Compact Support

Definition - Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define the support of φ to be the set

$$\operatorname{supp} \varphi = \overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}}$$ Indicates closure

Example:



$\operatorname{supp} \varphi = [a, b]$

Definition - We say $\varphi : \mathbb{R} \to \mathbb{R}$ has compact support if $\operatorname{supp} \varphi$ is bounded and we also say it is compactly supported. The vector space of smooth functions with compact support is denoted $C_0^\infty(\mathbb{R})$.

Proposition - If $\varphi \in C_0^\infty(\mathbb{R})$ then φ cannot be analytic.

proof:

$\varphi \in C_0^\infty(\mathbb{R})$ if and only if there exists $x_0 \in \partial \operatorname{supp}(\varphi)$ such that the Taylor series of φ about x_0 is given by

$$\varphi(x) = \varphi(x_0) + \varphi'(x_0)(x - x_0) + \tfrac{1}{2}\varphi''(x_0)(x - x_0)^2 + \ldots = 0$$

However, since $x_0 \in \partial \operatorname{supp}(\varphi)$ this is only true if $x = x_0$, i.e., the radius of convergence is 0.

Example:

If
$$\varphi(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$
then φ is a smooth function.

proof:

By construction $\lim_{x \to 0} e^{-1/x} = 0$ and thus φ is continuous. Now, assume
$$\frac{d^k \varphi}{dx^k} = \begin{cases} p_k(1/x)\, e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$
for some polynomial p_k and $k \in \mathbb{N}$. Therefore, for $x > 0$
$$\frac{d^{k+1}\varphi}{dx^{k+1}} = \left(\frac{-p_k'(1/x)}{x^2} + \frac{(1/x)}{x^2} \right) e^{-1/x} = p_{k+1}(1/x)\, e^{-1/x}$$
Thus, by the principal of mathematical induction for all $k \in \mathbb{N}$
$$\frac{d^k \varphi}{dx^k} = \begin{cases} p_k(1/x)\, e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$
Consequently, apply L'Hospital's rule iteratively we have that
$$\lim_{x \to 0^+} \frac{d^k \varphi}{dx^k} = 0$$
and thus φ is smooth. ■

Proposition - Whenever $a<b$, there is a function $\psi \in C_0^\infty(\mathbb{R})$ satisfying

$$\psi_{[a,b]}(x) = \begin{cases} 0, & x \le a \\ \in (0,1), & a<x<b \\ 0, & x \ge b \end{cases}$$

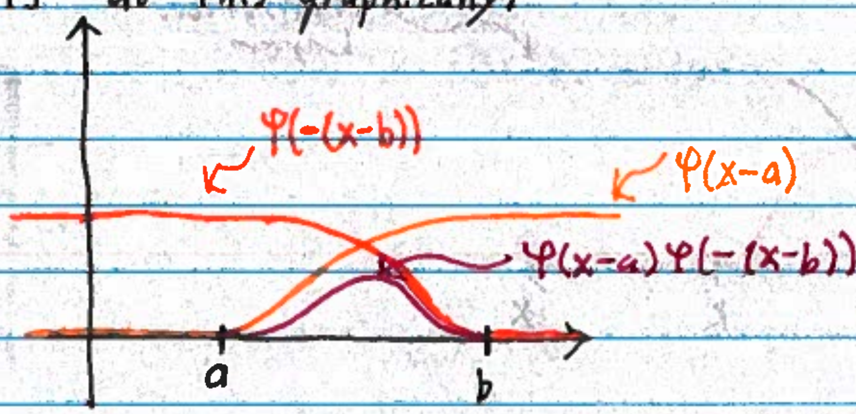
Such a function is called a bump function.

proof:

Let

$$\varphi(x) = \begin{cases} 0, & x \le 0 \\ e^{-1/x}, & x>0 \end{cases}$$

Lets do this graphically:



The function

$$\psi_{[a,b]}(x) = \varphi(x-a)\varphi(b-x)$$
$$= \begin{cases} 0, & x \le a \\ e^{-1/x-a} e^{-1/b-x}, & a<x<b \\ 0, & x \ge b \end{cases}$$

Works

Proposition – For $R>0$ there exist a smooth function $\rho_R(x)$ such that

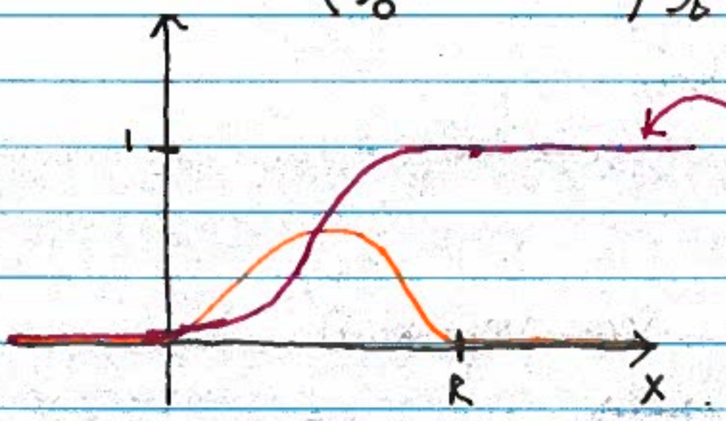
$$\rho_R(x) = \begin{cases} 0, & x \le 0 \\ \in (0,1), & 0<x<R \\ 1, & x \ge R \end{cases}$$

proof:

Let $\psi_{[0,R]}$ be a smooth bump function supported on $[0,R]$.

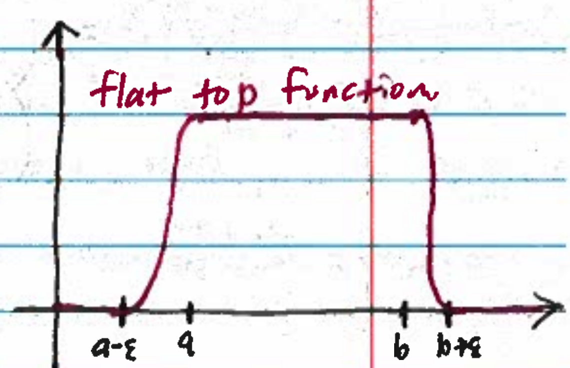
Let

$$\rho_R(x) = \left(\int_0^R \psi_{[0,R]}(x)\,dx \right)^{-1} \int_0^x \psi_{[0,R]}(s)\,ds$$



Proposition – For $a<b$ and $0<\varepsilon<<1$ there exists a smooth function $\eta^{\varepsilon}_{[a,b]}$ satisfying

$$\eta^{\varepsilon}_{[a,b]}(x) = \begin{cases} 0, & x \le a-\varepsilon \\ \in (0,1), & a-\varepsilon<x<a \\ 1, & a \le x \le b \\ \in (0,1), & b<x<b+\varepsilon \\ 0, & b+\varepsilon \le x \end{cases}$$



proof:

$\eta^{\varepsilon}_{[a,b]}(x) = \rho_\varepsilon(x-a)\,\rho_\varepsilon(b-x)$ works.