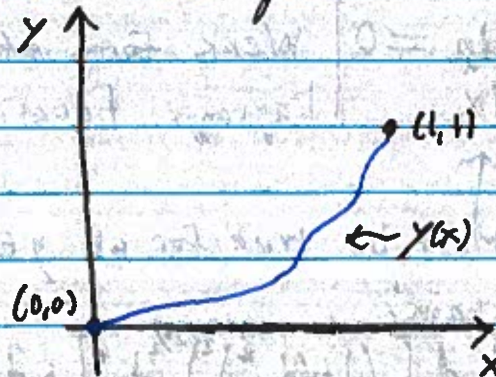
Lecture #11: Calculus of Variations

Example:

Find the curve connecting the point $(0,0)$ to $(1,1)$ that minimizes arclength.



If y is a smooth function of x satisfying $y(0)=0$ and $y(1)=0$

$$L[y(x)]=\int_0^1\sqrt{1+y'(x)^2}\,dx$$

Define

$$\mathcal{A}=\{y\in C^\infty([0,1]);\ y(0)=y(1)=0\}.$$

Then, $L:\mathcal{A}\to\mathbb{R}$ is a functional. Now suppose y^* is a minimum of L. Consequently, for all $\varepsilon>0$, if $\eta\in C_0^\infty([0,1])$ then $y^*+\varepsilon\eta\in\mathcal{A}$. Now, define a function

$$\begin{aligned} g(\varepsilon)&=L[y^*+\varepsilon\eta] \\ &=\int_0^1\sqrt{1+\left(\frac{dy^*}{dx}+\varepsilon\frac{d\eta}{dx}\right)^2}\,dx \end{aligned}$$

Since, y^* is a minimum, assuming g is differentiable, we have that

$$g(0)=L[y^*] \text{ and } g'(0)=0.$$

Differentiating, we have

$$g'(\varepsilon) = \frac{d}{d\varepsilon}\int_0^1 \sqrt{1+\left(\frac{dy^*}{dx}+\varepsilon\frac{d\eta}{dx}\right)^2}\,dx$$

$$= \int_0^1 \frac{1}{2}\left(1+\left(\frac{dy^*}{dx}+\varepsilon\frac{d\eta}{dx}\right)^2\right)^{-1/2} 2\left(\frac{dy^*}{dx}+\varepsilon\frac{d\eta}{dx}\right)\frac{d\eta}{dx}$$

$$\Rightarrow \boxed{g'(0) = \int_0^1 \left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{-1/2}\frac{dy^*}{dx}\frac{d\eta}{dx} = 0}$$

Weak form of the Euler-Lagrange Equations.

Must be true for all $\eta \in C_0^\infty([0,1])$

We can integrate by parts to obtain:

$$g'(0) = \underbrace{\cancel{\left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{-1/2}\frac{dy^*}{dx}\eta\Big|_0^1}}_{\text{Vanishes since } \eta\in C_0^\infty[0,1]} \to 0 \; - \int_0^1 \underbrace{\frac{d}{dx}\left[\left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{-1/2}\frac{dy^*}{dx}\right]}_{=0 \text{ since true for all } \eta}\eta\,dx = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[\left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{-1/2}\frac{dy^*}{dx}\right] = 0}$$

← Strong form of the Euler-Lagrange Equations.

$$\Rightarrow \left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{-1/2}\frac{dy^*}{dx} = C$$

$$\Rightarrow \frac{dy^*}{dx} = C\left(1+\left(\frac{dy^*}{dx}\right)^2\right)^{1/2}$$

$$\Rightarrow \left(\frac{dy^*}{dx}\right)^2 = C^2\left(1+\left(\frac{dy^*}{dx}\right)^2\right)$$

$$\Rightarrow \left(\frac{dy^*}{dx}\right)^2 = \frac{C^2}{1-C^2} = A^2$$

$$\Rightarrow \frac{dy^*}{dx} = A \quad \text{or} \quad \frac{dy^*}{dx} = -A$$

$$\Rightarrow y^* = Ax+B \quad \text{or} \quad y^* = -Ax+B$$

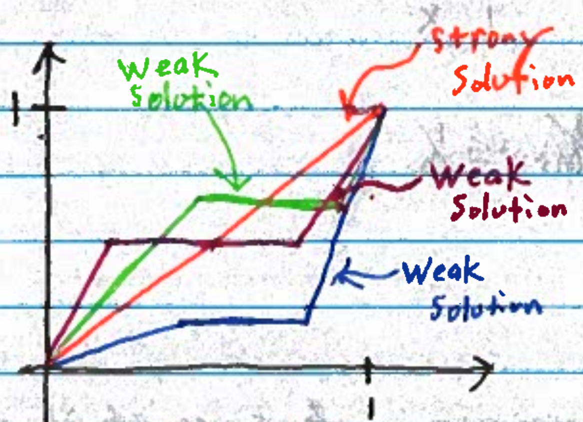
From boundary conditions

$$y = x.$$

This derivation gives us the intuitively correct answer but there were a number of assumptions we made:

1. We assume $y \in C^\infty$. However, all we really needed was that
$$\int_0^1 |y'(x)|\,dx < \infty.$$
2. When we integrated by parts, we implicitely assumed that y^* is second differentiable but, given item 1, this might not be the case.
3. We assumed there exists a smooth function with compact support.
4. We assumed that just because y^* is an extremizer that it must be a minimizer.
5. We assumed we could switch $\frac{d}{d\varepsilon}$ and the integral.

In particular, we can "glue" together strong solutions to form weak solutions that are also extremizers.



How can we tell which ones are actually minimizers? Resolving this issue is very challenging.

Example:

Consider the functional $L: \mathcal{A} \to \mathbb{R}$ defined by

$$L[y(x)] = \int_0^1 y(x)^2 dx + \int_0^1 (y'(x)^2 - 1)^2 dx,$$

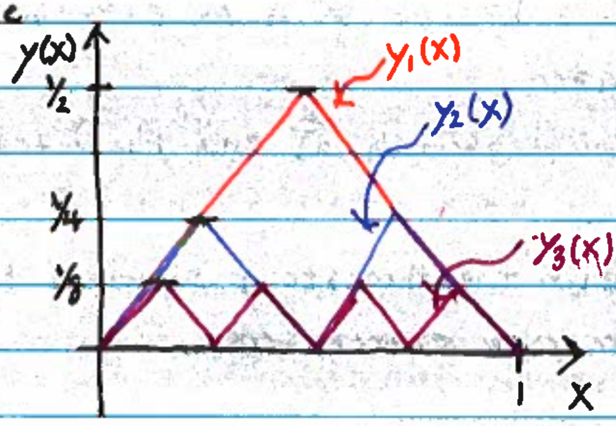
where

$$\mathcal{A} = \{ y(x) : \int_0^1 y(x)^4 dx + \int_0^1 y'(x)^4 dx < \infty \text{ and } y(0) = y(1) = 0 \}.$$

This functional has no minimizers.

proof:

By construction, for all $y \in \mathcal{A}$, $L[y] \geq 0$. In particular if $L[y] = 0$ then both $\int_0^1 y(x)^2 dx = 0$ and $\int_0^1 (y'(x)^2 - 1)^2 dx = 0$ which implies $y = 0$ and $y' = \pm 1$ which is a contradiction. However, consider the following sequence



By construction

$$L[y_n] = \int_0^1 y_n(x)^2 dx \leq n \cdot \frac{1}{(n+1)^2} \leq \frac{1}{n}.$$

Consequently,

$$\lim_{n \to \infty} L[y_n] = 0.$$

This proves there cannot be a minimizer in $\mathcal{A}$ since if it was a minimizer y^* we would need $L[y^*] = 0$.

Euler-Lagrange Equations and Conservation Laws

Find the strong extremizers of the functional:

$$I[y] = \int_a^b \underbrace{L(x, y, y')}_{\text{Lagrangian}} dx$$

Let $\eta \in C_0^{\infty}([a,b])$ and define

$$g_\eta(\varepsilon) = I[y^* + \varepsilon\eta]$$

Note, if $g_\eta(\varepsilon)$ is smooth

$$g_\eta(\varepsilon) = g_\eta(0) + g_\eta'(0)\varepsilon + o(\varepsilon)$$
$$= I[y^*] + g_\eta'(0)\varepsilon + o(\varepsilon)$$
$$\Rightarrow g_\eta(\varepsilon) - I[y^*] = g_\eta'(0)\varepsilon + o(\varepsilon)$$
$$\Rightarrow \frac{g_\eta(\varepsilon) - I[y^*]}{\varepsilon} = g_\eta'(0) + o(1)$$

We define the Gateaux derivative in the direction η by:

$$\boxed{\delta_\eta I = \frac{\delta I}{\delta \eta} = g_\eta'(0).}$$

For $I[y] = \int_a^b L(x, y^*, \frac{dy^*}{dx})dx$ we have:

$$g_\eta(\varepsilon) = \int_a^b L(x, y^* + \varepsilon\eta, \tfrac{dy^*}{dx} + \varepsilon\eta')dx$$
$$\Rightarrow g_\eta'(\varepsilon) = \frac{d}{d\varepsilon}\int_a^b L(x, y^* + \varepsilon\eta, \tfrac{dy^*}{dx} + \varepsilon\eta')dx$$
$$= \int_a^b \frac{d}{d\varepsilon} L(x, y^* + \varepsilon\eta, \tfrac{dy^*}{dx} + \varepsilon\eta')dx$$
$$= \int_a^b \left(\frac{\partial L}{\partial y}\bigg|_{y^*+\varepsilon\eta} \eta + \frac{\partial L}{\partial y'}\bigg|_{y^*+\varepsilon\eta} \eta' \right) dx$$
$$= \frac{\partial L}{\partial y'}\bigg|_{y^*+\varepsilon\eta} \eta \bigg|_a^b + \int_a^b \left(\frac{\partial L}{\partial y}\bigg|_{y^*+\varepsilon\eta} - \frac{d}{dx}\frac{\partial L}{\partial y'}\bigg|_{y^*+\varepsilon\eta} \right)\eta\, dx$$

$$\Rightarrow \delta_\eta I|_{y^*} = \frac{\delta I}{\delta \eta}\bigg|_{y^*} = \int_a^b \left(\frac{\partial L}{\partial y}\bigg|_{y^*} - \frac{d}{dx}\frac{\partial L}{\partial y'}\bigg|_{y^*} \right)\eta\, dx$$

The Euler-Lagrange Equations satisfied by a strong minimizer are:

$$\boxed{\left.\frac{\partial L}{\partial y}\right|_{y^*} - \frac{d}{dx}\left.\frac{\partial L}{\partial y'}\right|_{y^*} = 0}$$

Euler-Lagrange Equations.
Solutions are called extremizers.

Example:

$$I[y] = \int_0^1 (y'^2 + 3y + 2x)\,dx, \quad y(0)=0,\ y(1)=1.$$

$$\Rightarrow \frac{\partial L}{\partial y} = 3, \quad \frac{\partial L}{\partial y'} = 2y', \quad \frac{d}{dx} 2y' = 2y''$$

Therefore, the Euler-Lagrange equations are

$$3 - 2y'' = 0$$

$$y'' = 3/2$$

$$y(x) = 3/4 x^2 + c_1 x + c_2$$

Boundary conditions imply

$$y(x) = 3/4 x^2 + 1/4 x$$

Theorem - Suppose x^* is an extremizer of the functional

$$\int_{t_0}^{t_f} L(t, x, \dot{x})\,dt.$$

(i) If L is independent of x, then the following quantity is constant in time:

$$\left.\frac{\partial L}{\partial \dot{x}}\right|_{x^*}$$

(ii) If L is independent of t, then the following quantity is constant in time:

$$L(x^*, \dot{x}^*) - \dot{x}^* \left.\frac{\partial L}{\partial \dot{x}}\right|_{x^*}$$

proof:

(i) Since L is independent of x, $\frac{\partial L}{\partial x} = 0$ and thus the Euler-Lagrange equations are

$$\frac{d}{dt}\left.\frac{\partial L}{\partial \dot{x}}\right|_{x^*} = 0$$

$$\Rightarrow \left.\frac{\partial L}{\partial \dot{x}}\right|_{x^*} = \text{constant}$$

(ii) Differentiating we have that

$$\frac{d}{dt}\left[L(x^*,\dot{x}^*) - \dot{x}^*\frac{\partial L}{\partial \dot{x}}\Big|_{x^*}\right] = \frac{\partial L}{\partial x}\Big|_{x^*}\dot{x}^* + \frac{\partial L}{\partial \dot{x}}\Big|_{x^*}\ddot{x}^* - \ddot{x}^*\frac{\partial L}{\partial \dot{x}}\Big|_{x^*} - \dot{x}^*\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^*}\Big|,$$

$$= \left(\frac{\partial L}{\partial x}\Big|_{x^*} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\Big|_{x^*}\right)\dot{x}^* = 0.$$

■

Definition - For the functional $I[x] = \int_{t_0}^{t_f} L(x,\dot{x})\,dt$ we call the conserved quantity

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$$

the Hamiltonian

Example:

Let $V:\mathbb{R}\to\mathbb{R}$ be a smooth function and consider the following functional $I[x] = \int_{t_0}^{t_f}\left(\frac{1}{2}\dot{x}^2 - V(x)\right)dx.$

$$\Rightarrow \frac{\partial L}{\partial x} = -V'(x),\quad \frac{\partial L}{\partial \dot{x}} = \dot{x},\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x}$$

The Euler-Lagrange equations are thus

$$-V'(x) - \ddot{x} = 0$$

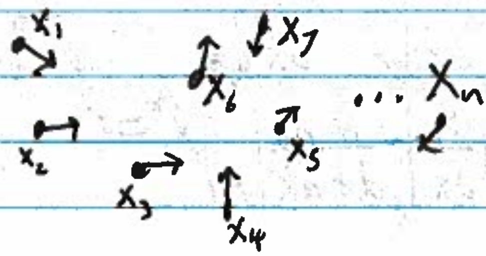
$$\Rightarrow \ddot{x} = -V'(x)$$ (Hamiltonian system / conservative system).

The conserved quantity is

$$L(x,\dot{x}) - \dot{x}\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}\dot{x}^2 - V(x) - \dot{x}^2 = -\frac{1}{2}\dot{x}^2 - V(x)$$

$$\Rightarrow H = \frac{1}{2}\dot{x}^2 + V(x)$$

Euler-Lagrange Equations in Higher Dimensions



Consider n-particles whose motion extremizes the following functional

$$I[(x_1,\dots,x_n)] = \int_{t_0}^{t_f} L(t, x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n)\,dt$$

Perturb in the x_i direction by a smooth function with compact support η_i

$$\delta_{\eta_i} I = \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} I[(x_1^*,\dots,x_i^*+\varepsilon\eta_i,\dots,x_n^*)]$$

$$= \lim_{\varepsilon\to 0} \int_{t_0}^{t_f} \left(\left.\frac{\partial L}{\partial x_i}\right|_{x_i^*+\varepsilon\eta_i} \eta_i + \left.\frac{\partial L}{\partial \dot{x}_i}\right|_{\dot{x}_i^*+\varepsilon\dot{\eta}_i} \dot{\eta}_i \right) dt$$

$$= \int_{t_0}^{t_f} \left.\left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \right)\right|_{x_i^*} \eta_i \, dt$$

$\Rightarrow$ The governing equations for the n-particles is the system of equations:

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0}$$