Lecture #12: Crash Course on Tensors

Index Notation

A vector $\vec{v} \in \mathbb{R}^n$ can be represented as

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{or} \quad \vec{v} = (v_1, \dots, v_n), \quad \text{or as} \quad \vec{v} = v_i$$

free index, stand in for all possible values of i.

A tensor (rank m) is a multilinear map $T: \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ times}} \to \mathbb{R}$

That is for all $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$, $\vec{u} \in \mathbb{R}$, and $\lambda \in \mathbb{R}$

$$T(\vec{v}_1, \dots, \vec{v}_i + \lambda \vec{u}, \dots, \vec{v}_n) = T(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) + \lambda T(\vec{v}_1, \dots, \vec{u}, \dots, \vec{v}_n).$$

Example:

Let $A \in \mathbb{R}^{n \times n}$. Define $T: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$T(\vec{u}, \vec{v}) = \langle \vec{u}, A\vec{v} \rangle$$

T is a tensor since

$$\begin{aligned}
T(\vec{u}, \vec{v} + \lambda \vec{w}) &= \langle \vec{u}, A(\vec{v} + \lambda \vec{w}) \rangle \\
&= \langle \vec{u}, \sum_{j=1}^{n} (A_{ij}(v_j + \lambda w_j)) \vec{e}_i \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i A_{ij} (v_j + \lambda w_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i A_{ij} v_j + \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} u_i A_{ij} w_j \\
&= \langle \vec{u}, A\vec{v} \rangle + \lambda \langle \vec{u}, A\vec{w} \rangle = T(\vec{u}, \vec{v}) + \lambda T(\vec{u}, \vec{w})
\end{aligned}$$

$$\begin{aligned}
T(\vec{u} + \lambda \vec{w}, \vec{v}) &= \langle \vec{u} + \lambda \vec{w}, A\vec{v} \rangle \\
&= \langle \vec{u} + \lambda \vec{w}, \sum_{j=1}^{n} A_{ij} v_j \vec{e}_i \rangle \\
&= \sum_{i=1}^{n} (u_i + \lambda w_i) \sum_{j=1}^{n} A_{ij} v_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (u_i + \lambda w_i) A_{ij} v_j
\end{aligned}$$

$$\Rightarrow T(\vec{u}+\lambda\vec{w},\vec{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i A_{ij} v_j + \lambda \sum_{i=1}^{n}\sum_{j=1}^{n} w_i A_{ij} v_j$$

$$= \langle \vec{u}, A\vec{v}\rangle + \lambda\langle \vec{w}, A\vec{v}\rangle = T(\vec{u},\vec{v}) + \lambda T(\vec{w},\vec{v})$$

Consequently, T is a tensor.

Components

Suppose $T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is a rank m tensor. If $\vec{v}^1, \ldots, \vec{v}^m$ are vectors of the form

$$\begin{aligned} \vec{v}^1 &= v_1^1 \vec{e}_1 + \ldots + v_n^1 \vec{e}_n \\ &\vdots \\ \vec{v}^m &= v_1^m \vec{e}_1 + \ldots + v_n^m \vec{e}_n \end{aligned}$$

$$\Rightarrow T(\vec{v}^1, \ldots, \vec{v}^m) = T(v_1^1\vec{e}_1 + \ldots + v_n^1\vec{e}_n, \ldots, v_1^m\vec{e}_1 + \ldots + v_n^m\vec{e}_n)$$

$$= \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} v_{i_1}^1 v_{i_2}^2 \cdots v_{i_m}^m \underbrace{T(\vec{e}_{i_1}, \vec{e}_{i_2}, \ldots, \vec{e}_{i_m})}_{A_{i_1 i_2 \ldots i_m}}$$

$\rightarrow$ Components of tensor. Computed by action on basis vectors.

Summation Notation

The previous calculations were extremely cumbersome. There are a number of redundancies that can be removed.

1. The summation signs can be removed by assuming a summation sign when indices are repeated.
2. The basis vectors $\vec{e}_i$ can be removed from calculations and just replaced with components.

Example:

In $\mathbb{R}^n$, every rank 1 tensor corresponds to a vector $\vec{V} \in \mathbb{R}^n$.

proof:

Let T be a rank 1 tensor with components $V_i = T(e_i)$.

Consequently, for $\vec{W} = w_1\vec{e}_1 + \dots w_n\vec{e}_n$ we have

$$T(\vec{W}) = V_i W_i \longleftarrow \text{Summation Notation}$$

$$= \langle \vec{V}, \vec{W} \rangle.$$

Thus $T(\vec{W})$ can be put into one-to-one correspondence with a vector $\vec{V}$ through the action

$$T(\vec{W}) = \langle \vec{V}, \vec{W} \rangle$$

Example:

In $\mathbb{R}^n$, every rank 2 tensor corresponds to a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^n$.

proof:

Let T be a rank two tensor with components $A_{ij} = T(\vec{e}_i, \vec{e}_j)$.

Consequently, for vectors $\vec{V}$ and $\vec{W}$ we have

$$T(\vec{V}, \vec{W}) = T_{ij} V_j W_i \leftarrow \text{Summation notation}$$

If we let A be the matrix with components T_{ij} we have that

$$T(\vec{v}, \vec{w}) = \langle \vec{w}, A\vec{v} \rangle$$

Special Symbols

1. $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ (Kronecker Delta)

$$\Rightarrow \langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$$

$$\Rightarrow \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

2. Alternating tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases}$$

$\varepsilon_{123} = 1$ (0 per.)

$\varepsilon_{231} = 1$ (2 per)

$\varepsilon_{321} = -1$ (3 per)

$\varepsilon_{332} = 0$

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$ then

$$c_i = \varepsilon_{ijk} v_j w_k$$

is the cross product:

$$\vec{c} = \vec{v} \times \vec{w}.$$

Example:

The area of a parrelogram spanned by two vectors $\vec{u}$ and $\vec{v}$ is

$$\text{Area} = \| \vec{a} \times \vec{b} \|$$

Express in summation notation:

$$c_i = \varepsilon_{ijk} a_j b_k$$

$$\Rightarrow \| \vec{c} \|^2 = c_i c_i$$

$$= \varepsilon_{ijk} a_j b_k \varepsilon_{ijk} a_m b_n$$

$$\Rightarrow \text{Area} = \sqrt{\varepsilon_{ijk} a_j b_k \varepsilon_{ijk} a_m b_n}$$

Differentiation

Differentiation with respect to time

$\vec{X} \rightarrow$ position with respect to time in $\mathbb{R}^n$

$\searrow$ X_i (subscript notation)

$$\frac{d\vec{x}}{dt} = \dot{\vec{x}} = \dot{x}_i = x_{i,t}$$

$$\frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}} = \ddot{x}_i = x_{i,tt}$$

Differentiation with respect to space

1. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = f_{,i} \leftarrow \text{Vector field}$$

2. Suppose $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\nabla F = \vec{F}_{i,j} \leftarrow \text{Rank 2 tensor field (Jacobian)}$$

3. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\nabla^2 f = f_{,ij} \leftarrow \text{Rank 2 tensor field (Hessian)}$$

4. Taylor's theorem for $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(\vec{x}) = f(\vec{x}^*) + f_{,i}(\vec{x}^*)(x_i - x_i^*) + \tfrac{1}{2} f_{,ij}(\vec{x}^*)(x_i - x_i^*)(x_j - x_j^*)$$
$$+ \tfrac{1}{6} f_{,ijk}(\vec{x}^*)(x_i - x_i^*)(x_j - x_j^*)(x_k - x_k^*) + \ldots$$

5. Taylor's theorem for $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$F_i(\vec{x}) = F_i(\vec{x}^*) + F_{i,j}(\vec{x}^*) + \tfrac{1}{2} F_{i,jk}(\vec{x}^*)(x_j - x_j^*)(x_k - x_k^*) + \ldots$$

6. If $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ then

$$\nabla \cdot \vec{F} = F_{i,i}$$

7. If $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\nabla \times \vec{F} = \varepsilon_{ijk} F_{j,k}$$

8. If $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$$\Delta f = f_{,ii}$$