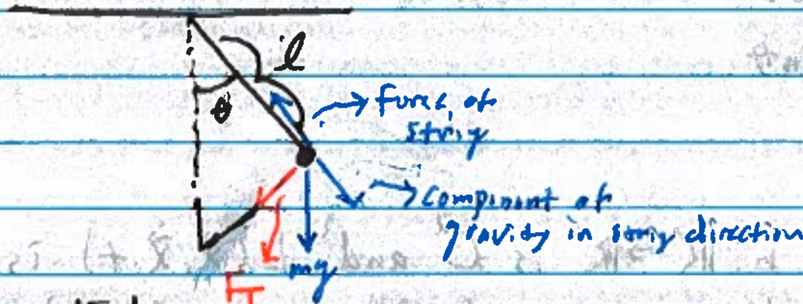
Lecture 13: Lagrangian Mechanics

Example:

What are equations of motion of simple pendulum of mass m:



Classic Approach

$$\sin\theta = \frac{|F_T|}{mg}$$

$$\Rightarrow |F_T| = mg\sin\theta$$

If we let s denote the arclength along the path we have

$$m\ddot{s} = -mg\sin\theta$$

$$s = l\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta \leftarrow \text{Equations of motion.}$$

What about the Lagrangian approach??

$$L = \underbrace{\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2}_{\text{Kinetic energy}} - \underbrace{mgy}_{\text{potential energy}}$$

Convert to new generalized coordinate θ

$$x = \ell\sin\theta$$

$$y = \ell\cos\theta$$

$$\Rightarrow \dot{x} = \cancel{-}\ell\cos\theta\,\dot{\theta}$$

$$\dot{y} = -\ell\sin\theta\,\dot{\theta}$$

$$\Rightarrow L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\sin\theta$$

The dynamics should extremize the following functional

$$I[\theta] = \int_{t_1}^{t_2} \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell\cos\theta$$

$$\Rightarrow \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mg\sin\theta - \frac{d}{dt} m\ell^2\dot{\theta} = -mg\sin\theta - m\ell^2\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$$

Theorem: Suppose $h: \mathbb{R}^n \to \mathbb{R}^n$ is C^2 and $L(\vec{x}, \dot{\vec{x}}, t)$ is a Lagrangian. The dynamics of $\vec{y} \in \mathbb{R}^n$ where $\vec{x} = h(\vec{y})$ is given by the Euler-Lagrange equations of the Lagrangian

$$\tilde{L}(\vec{y}, \dot{\vec{y}}, t) = L(h(\vec{y}), \nabla h(\vec{y})\dot{\vec{y}}, t).$$

proof:

I am going to first make some observation on notation. In tensor notation we have:

$$\nabla h(\vec{y})\dot{\vec{y}} = h_{j,k}\dot{y}_k \quad \text{and} \quad h(\vec{y}) = h_j$$

Therefore,

$$(i)\ \frac{\partial \tilde{L}}{\partial \dot{y}_i} = \frac{\partial}{\partial \dot{y}_i} L(h(\vec{y}), \nabla h(\vec{y})\dot{\vec{y}}, t)$$

$$= \frac{\partial}{\partial \dot{x}_m} L(\vec{x}, \dot{\vec{x}}, t) \frac{\partial}{\partial \dot{y}_i} h_{m,k}\dot{y}_k$$

$$= \frac{\partial}{\partial \dot{x}_m} L(\vec{x}, \dot{\vec{x}}, t) h_{m,k} \delta_{ki}$$

$$= \frac{\partial}{\partial \dot{x}_m} L(\vec{x}, \dot{\vec{x}}, t) h_{m,i}$$

$$= \frac{\partial}{\partial \dot{x}_j} L(\vec{x}, \dot{\vec{x}}, t) h_{j,i}$$

(ii) $\frac{\partial \tilde{L}}{\partial y_i} = \frac{\partial}{\partial y_i} L(h(\vec{y}), \nabla h(\vec{y})\dot{\vec{y}}, t)$

$$= \frac{\partial}{\partial x_m} L(\vec{x},\dot{\vec{x}},t)\frac{\partial}{\partial y_i} h_m + \frac{\partial}{\partial \dot{x}_m} L(\vec{x},\dot{\vec{x}},t)\frac{\partial}{\partial y_i} h_{m,k}\dot{y}_k$$

$$= \frac{\partial}{\partial x_m} L(\vec{x},\dot{\vec{x}},t)h_{m,i} + \frac{\partial}{\partial \dot{x}_m} L(\vec{x},\dot{\vec{x}},t)h_{m,ki}\dot{y}_k$$

(iii) $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{y}_i} = \frac{d}{dt}\left(\frac{\partial}{\partial x_j} L(\vec{x},\dot{\vec{x}},t)h_{j,i}\right)$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)\right)h_{j,i} + \frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)h_{j,ik}\dot{y}_k$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)\right)h_{j,i} + \frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)h_{j,ki}\dot{y}_k$$

$$\Rightarrow \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{y}_i} - \frac{\partial \tilde{L}}{\partial y_i} = \frac{d}{dt}\left(\frac{\partial}{\partial x_j} L(\vec{x},\dot{\vec{x}},t)\right)h_{j,i} + \cancel{\frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)h_{j,ki}\dot{y}_k}$$

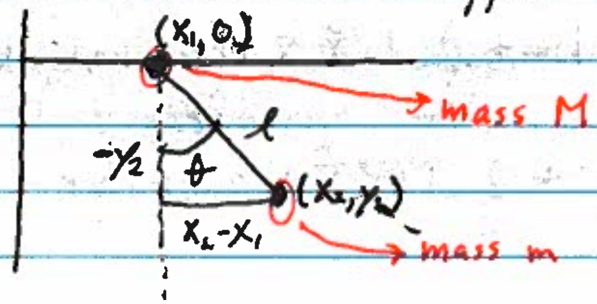
$$- \frac{\partial}{\partial x_m} L(\vec{x},\dot{\vec{x}},t)h_{m,i} - \cancel{\frac{\partial}{\partial \dot{x}_j} L(\vec{x},\dot{\vec{x}},t)h_{j,ki}\dot{y}_k}$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial x_j} - \frac{\partial L}{\partial x_j}\right)h_{j,i}$$

$$= 0.$$

Example:

Pendulum attached to a support that can move horizontally



$$y_2 = -\ell\cos\theta$$

$$x_2 = x_1 + \ell\sin\theta$$

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m((\dot{x}_1 + \ell\cos\theta\,\dot{\theta})^2 + \ell^2\sin^2\theta\,\dot{\theta}^2)$$

$$= \frac{1}{2}(M+m)\dot{x}_1^2 + m\ell\cos\theta\,\dot{x}_1\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$V = mgy_2 = -mg\ell\cos\theta$$

$$\Rightarrow L = T - V = \frac{1}{2}(M+m)\dot{x}_1^2 + m\ell\cos\theta\,\dot{x}_1\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \frac{d}{dt}((M+m)\dot{x}_1 + m\ell\cos\theta\,\dot{\theta}) = 0$$

$$\Rightarrow \boxed{(M+m)\ddot{x}_1 + m\ell\cos\theta\,\ddot{\theta} - m\ell\sin\theta\cdot\dot{\theta}^2 = 0}$$

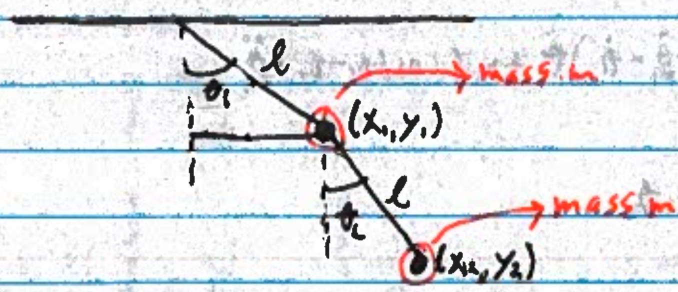
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(m\ell\cos\theta\,\dot{x}_1 + m\ell^2\dot{\theta}) + mg\ell\sin\theta$$

$$\Rightarrow \boxed{-m\ell\sin\theta\,\dot{\theta}\dot{x}_1 + m\ell\cos\theta\,\ddot{x}_1 + m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0}$$

$$\Rightarrow \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} ml\sin\theta \cdot \dot{\theta}^2 \\ ml\sin\theta\,\dot{\theta}\dot{x}_1 - mgl\sin\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \frac{1}{(M+m)ml^2 - m^2l^2\cos^2\theta} \begin{bmatrix} ml^2 & -ml\cos\theta \\ -ml\cos\theta & M+m \end{bmatrix} \begin{bmatrix} ml\sin\theta\,\dot{\theta}^2 \\ ml\sin\theta\,\dot{\theta}\dot{x}_1 - mgl\sin\theta \end{bmatrix}$$

Example:



$$x_1 = l\sin\theta_1 \qquad x_2 = l\sin\theta_1 + l\sin\theta_2$$

$$y_1 = -l\cos\theta_1 \qquad y_2 = -l\cos\theta_1 - l\cos\theta_2$$

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{y}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}m\dot{y}_2^2 \\ &= \tfrac{1}{2}ml^2\dot{\theta}_1^2 + \tfrac{1}{2}m\left((l\cos\theta_1\dot{\theta}_1 + l\cos\theta_2\dot{\theta}_2)^2 + (l\sin\theta_1\dot{\theta}_1 + l\sin\theta_2\dot{\theta}_2)^2\right) \\ &= \tfrac{1}{2}ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml^2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + ml\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 \\ &= ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml^2(\cos(\theta_1-\theta_2))\dot{\theta}_1\dot{\theta}_2 \end{aligned}$$

$$\begin{aligned} V &= -ml\cos\theta_1 - ml\cos\theta_1 - ml\cos\theta_2 \\ &= -2ml\cos\theta_1 - ml\cos\theta_2 \end{aligned}$$

$$\Rightarrow L = ml^2\dot{\theta}_1^2 + \tfrac{1}{2}ml^2\dot{\theta}_2^2 + ml(\cos(\theta_1-\theta_2))\dot{\theta}_1\dot{\theta}_2 + 2ml\cos\theta_1 + ml\cos\theta_2$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}_1} = 2ml^2\dot{\theta}_1 + ml(\cos(\theta_1-\theta_2))\theta_1\dot{\theta}_2$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = 2ml^2\ddot{\theta}_1 - ml\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_2 + ml\cos(\theta_1-\theta_2)\ddot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_1} = -ml\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2ml\sin\theta_1$$

$$\Rightarrow \boxed{2ml^2\ddot{\theta}_1 - ml\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_2 + ml\cos(\theta_1-\theta_2)\ddot{\theta}_2 + ml\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + 2ml\sin\theta_1 = 0}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = ml^2\dot{\theta}_2 + ml\cos(\theta_1-\theta_2)\dot{\theta}_1$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = ml^2\ddot{\theta}_2 - ml\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_1 + ml\cos(\theta_1-\theta_2)\ddot{\theta}_1$$

$$\frac{\partial L}{\partial \theta_2} = ml\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - ml\sin\theta_2$$

$$\Rightarrow \boxed{ml^2\ddot{\theta}_2 - ml\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\dot{\theta}_1 + ml\cos(\theta_1-\theta_2)\ddot{\theta}_1 - ml\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + ml\sin\theta_2 = 0}$$

Conjugate Momentum

The Euler-Lagrange equations are very complicated. Is there a way to simplify the equations?

Suppose $L = \frac{1}{2}M_{ij}(\vec{x})\dot{x}_i\dot{x}_j - V(\vec{x})$. We know that

$$\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = \frac{\partial L}{\partial x_k}$$

This can be interpreted as an evolution equation for

$$p_k = \frac{\partial L}{\partial \dot{x}_k}$$

The quantities $p_K = \frac{\partial L}{\partial \dot{x}_K}$ are called the conjugate momentum.
Differentiating, we have that

$$p_K = \frac{\partial L}{\partial \dot{x}_K} = \frac{\partial}{\partial \dot{x}_K} \frac{1}{2} M_{ij} \dot{x}_i \dot{x}_j$$

$$= \frac{1}{2} M_{ij} \delta_{iK} \dot{x}_j + \frac{1}{2} M_{ij} \dot{x}_i \delta_{jK}$$

$$= \frac{1}{2} M_{Kj} \dot{x}_j + \frac{1}{2} M_{iK} \dot{x}_i$$

$$= \frac{1}{2} M_{Kj} \dot{x}_j + \frac{1}{2} M_{Ki} \dot{x}_i$$

$$= M_{Kj} \dot{x}_j$$

We also have that:

$$\dot{p}_K = \frac{\partial L}{\partial x_K} = -\frac{\partial V}{\partial x_K}$$

The governing equations are therefore:

$$\boxed{\begin{array}{ll} \dot{x}_j = M^{-1}_{jK} p_K & \Rightarrow \quad \dot{\vec{x}} = M^{-1} \vec{p} \\ \dot{p}_K = -\dfrac{\partial V}{\partial x_K} & \quad \dot{\vec{p}} = -\nabla_{\vec{x}} V \end{array}}$$

Return to Double Pendulum

$$T = m\ell^2 \dot{\theta}_1^2 + \tfrac{1}{2} m\ell^2 \dot{\theta}_2^2 + m\ell^2 (\cos(\theta_1 - \theta_2)) \dot{\theta}_1 \dot{\theta}_2$$

$$\Rightarrow M = \begin{bmatrix} m\ell^2 & \frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) & \frac{1}{2} m\ell^2 \end{bmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{\frac{1}{2} m^2 \ell^4 - \frac{1}{4} m^2 \ell^4 \cos^2(\theta_1 - \theta_2)} \begin{bmatrix} \frac{1}{2} m\ell^2 & -\frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) \\ -\frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) & m\ell^2 \end{bmatrix}$$

$$= \frac{4}{m^2 \ell^4 (1 + \sin^2(\theta_1 - \theta_2))} \begin{bmatrix} \frac{1}{2} m\ell^2 & -\frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) \\ -\frac{1}{2} m\ell^2 \cos(\theta_1 - \theta_2) & m\ell^2 \end{bmatrix}$$

$$V = -2mg\cos\theta_1 - mg\cos\theta_2$$

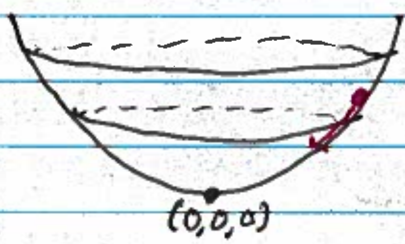
$$\Rightarrow \nabla_{\vec{\theta}} V = \begin{bmatrix} 2m\ell \sin\theta_1 \\ m\ell \sin\theta_2 \end{bmatrix}$$

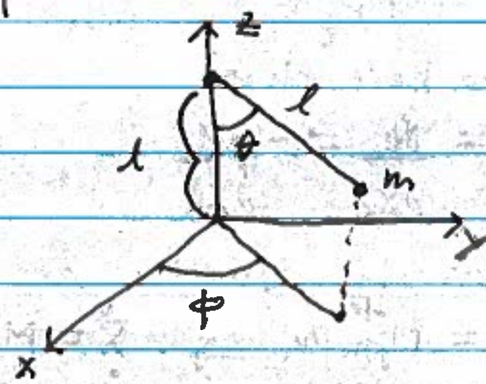
Therefore,

$$\boxed{\begin{aligned} \dot{\vec{\theta}} &= \frac{4}{m^2\ell^4(1+\sin^2(\theta_1-\theta_2))}\begin{bmatrix} \frac{1}{2}m\ell^2 & -\frac{1}{2}m\ell^2\cos(\theta_1-\theta_2) \\ -\frac{1}{2}m\ell^2\cos(\theta_1-\theta_2) & m\ell^2 \end{bmatrix}\vec{p} \\ \dot{\vec{p}} &= \begin{bmatrix} -2m\ell g\sin\theta_1 \\ -m\ell g\sin\theta_2 \end{bmatrix} \end{aligned}}$$

Example:

Particle in a spherical bowl





$$T = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

$$V = mg\ell(1-\cos\theta)$$

$$M = \begin{bmatrix} \frac{1}{2}m\ell^2 & 0 \\ 0 & \frac{1}{2}m\ell^2\sin^2\theta \end{bmatrix}$$

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mg\ell(1-\cos\theta)$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{1}{m\ell^2}p_\theta$$

$$p_\phi = m\ell^2\dot{\phi}\sin^2\theta \Rightarrow \dot{\phi} = \frac{1}{m\ell^2\sin^2\theta}p_\phi$$

$$\frac{\partial L}{\partial\theta} = -mg\ell\sin\theta$$

$$\frac{\partial L}{\partial\phi} = 0$$

The governing equations are therefore

$$\dot{\theta} = \frac{1}{ml^2} p_\theta$$

$$\dot{\phi} = \frac{1}{ml^2} p_\phi$$

$$\dot{p}_\theta = -mgl\sin\theta$$

$$\dot{p}_\phi = 0$$